PHYSICS DEPARTMENT
PRINCETON UNIVERSITY

GRADUATE GENERAL EXAMINATION

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III.

This part of the General Examination poses three questions on Condensed Matter Physics and three on Elementary Particles and Nuclear Physics. Answer three questions, at least one from each section.

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem name.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section A. Condensed Matter

1. Consider a chain of identical atoms of mass M. Each atom interacts with its nearest neighbors, and the Hamiltonian is written as

$$H = \sum_{j=1}^{\infty}[\frac{p_j^2}{2m} + U(x_j - x_{j-1} - a)],$$

where p_j and x_j are the momentum and coordinate of the jth atom, with $x_0 = 0$.

Define $y_j \equiv (x_j - x_{j-1}) - a$. The interaction potential $U(y)$ has a minimum at $y = 0$ so that the minimum of the classical energy corresponds to $x_j = ja$. Thus, y_j is the deviation of the spacing between two neighboring atoms from the minimal configuration.

a) Assume first that the potential is harmonic: $U(y) = Ay^2$. Determine the sound velocity v_s and the Debye temperature $k\theta_D = \hbar\omega_D$.

b) Using equipartition of energy among the degrees of freedom, evaluate the mean squared fluctuations $\langle y^2 \rangle$ as a function of temperature T. Determine high T and low T asymptotics.

c) Define $\hat{x}_j$ as the average deviation of the absolute position of the jth atom from its ideal location: $\hat{x}_j \equiv x_j - ja$. Find $\langle \hat{x}_j \rangle$ and its mean squared value, $\langle \hat{x}_j^2 \rangle$. What is the qualitative difference between these results and the case when *noninteracting* atoms are positioned at minima of a periodic potential $U(x) =$

This problem is continued on the next page.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section A. Condensed Matter (continued)

Problem 1. (continued)

$U(x+a)$? In which of the two cases, if either, should one expect to observe Bragg peaks in X-ray and neutron scattering?

d) Returning to the original Hamiltonian, let us now add a weak cubic perturbation to the harmonic interaction potential: $U(y) = Ay^2 + \frac{g}{3}y^3$, where g is very small. Calculate $\langle y_j \rangle$ and $\langle \hat{x}_j \rangle$ in this case. Evaluate the coefficient of thermal expansion α, and find the relation between α and the specific heat of the system. (Hint: you may approximate y_j^3 as $3y_j\langle y_j^2 \rangle$.)

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section A. Condensed Matter (continued)

2. The simplest classical model of magnetism is the Ising model, which describes the interaction of one-component spins σ_i located at the sites of a lattice. It is usual to consider only the two-spin interactions. However it is possible to include interactions of larger numbers of spins. Consider a Hamiltonian:

$$H = -h\sum_{i=1}^{N}\sigma_i - \frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - \frac{1}{12}\sum_{i,j,k,l=1}^{N} U_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l.$$

Here matrix elements J_{ij} and U_{ijkl} determine the two-spin and four-spin interactions, and h denotes an external magnetic field. $N \gg 1$ is the total number of lattice sites. To simplify the problem let us make the drastic assumption that both interactions are independent of the spin locations:

$$J_{ij} = \frac{J}{N}; \qquad U_{ijkl} = \frac{U}{N^3}.$$

a) Express the energy of each configuration in terms of the average magnetization

$$m = \frac{1}{N}\sum_{i=1}^{N}\sigma_i.$$

Present the effective field h_{eff} which acts on every spin, in terms of the magnetization and parameters h, J, and U. Use this connection between h_{eff} and m to determine the free energy $F(m, T, h)$, out to fourth order in m. Evaluate the entropy of the system, S.

This problem is continued on the next page.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section A. Condensed Matter (continued)

Problem 2. (continued)

b) For $h = 0$ and $U = 0$ determine the critical temperature below which spontaneous magnetization appears. Evaluate the temperature dependence of the spontaneous magnetization, $m(T)$, at a temperature close to the critical one.

c) Had you calculated the free energy to *sixth* order in m, you would have found the coefficient of m^6 to be $7N/60$. Using this information, sketch the phase diagram in the (T, U) plane in the vicinity of the point $U = k_B T = J$ and identify the phases (magnetic or nonmagnetic) and the orders of any phase transitions.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section A. Condensed Matter (continued)

3. The valence band in semiconductors with cubic symmetry (Si, Ge, GaAs, etc.) is complex. It consists of two subbands: *heavy* and *light* holes, which are degenerate at zero momentum, p. In the parabolic approximation, the dispersion laws of the excitations can be written as

$$E_h(p) = \frac{1}{2m_h}p^2 \quad \text{and} \quad E_l(p) = \frac{1}{2m_l}p^2,$$

where m_h and m_l denote the masses of heavy and light holes respectively. Let a bulk sample of such a semiconductor be doped by some acceptors with a concentration N, and studied at low temperature, T. Assume that all acceptors are ionized and that there are neither neutral defects nor compensation. You may also neglect interactions between the holes.

a) Determine the concentrations, N_h and N_l, of heavy and light holes.

b) When can the scattering of both types of charge carriers by ionized acceptors be considered in the Born approximation? Express your answer in terms of the concentration N, the dielectric constant ϵ of the semiconductor, and the masses.

c) Using this approximation evaluate *transport* scattering cross-sections, σ_h and σ_l, and mean free paths, ℓ_h and ℓ_l, for both types of carriers. (Do not worry about numerical factors. You may even use dimensional analysis).

This problem is continued on the next page.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section A. Condensed Matter (continued)

Problem 3. (continued)

d) Determine the residual ($T = 0$) resistivity of such a sample. Also determine the Hall voltage, V_H, that appears at zero temperature as a function of current, J, and magnetic field, H, perpendicular to the current.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section B. Elementary Particles and Nuclear Physics

1. The SuperKamiokande experiment sees evidence for flavor oscillations in neutrinos produced in the atmosphere by cosmic rays. One interpretation is that the deficit in observed muon neutrinos is the result of $\nu_\mu \to \nu_\tau$ mixing. Using a two-generation mixing model, the $\nu_\mu \to \nu_\tau$ mixing probability is given by

$$P(\nu_\mu \to \nu_\tau) = \sin^2(2\theta)\ \sin^2\left(\frac{\delta m^2 L}{4E}\right)$$

where θ is the mixing angle between flavor and mass eigenstates, δm^2 is the difference between the masses squared of the mass eigenstates, E is the neutrino energy, and L is the distance from the source, in units where $\hbar = c = 1$. The term $\sin^2(2\theta)$ is referred to as the amplitude of the oscillations. SuperK's results, interpreted in this model, give $\delta m^2 \approx 3 \times 10^{-3}$ eV^2, $\sin^2(2\theta) \approx 1$.

However, there are **three** known generations of neutrinos. Thus, in general, mixing will be governed by a 3×3 mixing matrix

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}.$$

This general case requires a great deal of experimental information to pin down, but we can make inferences about specific models. Including solar neutrino results suggests a model where two of the neutrino masses are very close together, and the third is larger:

$$m_3 = M > m = m_1 = m_2,$$

This problem is continued on the next page.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section B. Elementary Particles and Nuclear Physics (continued)

Problem 1. (continued)

with SuperK's $\delta m^2 = M^2 - m^2$. In this problem, assume this mass spectrum and also assume that there is no CP violation.

a) What does the assumption of no CP violation tell us about the matrix U?

b) Derive the probability of $\nu_\mu \to \nu_\tau$ as a function of distance from the source. (Hint: it helps to put in our mass model early in the calculation.)

c) When interpreted as two-generation mixing, SuperK measures $\sin^2(2\theta) \approx 1$. In our model of three-generation mixing, what is this a measurement of?
To allow a definite result, assume that SuperK measures $\sin^2(2\theta) = 0.98$ and that $U_{\tau 3} = 1/\sqrt{2}$. What is the amplitude of ν_μ oscillations to **electron** neutrinos?

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section B. Elementary Particles and Nuclear Physics (continued)

2. One proposed future accelerator is the muon collider. Such a machine would accelerate counter-rotating beams of μ^+ and μ^- to TeV-scale energies and bring them into collision. The machine could also be the source of a high intensity neutrino beam for a long-baseline neutrino oscillation experiment.

Among the technical challenges in designing a muon collider is dealing with the potential hazard of *neutrino* radiation. Consider the muon beam as it traverses a straight section of the accelerator. The radiation will produce a "hot spot" downstream from this straight section. In this problem you are asked to perform some calculations to estimate this radiation.

For your calculations, assume that the muon beam circulates around the machine with circumference s, and that this beam is made up of N muons.

The muon mass is 100 MeV/c^2 and the muon lifetime is 2 μs.

a) Indicate the dominant decay mode(s) of the μ^- and draw the Feynman diagram(s).

This problem is continued on the next page.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section B. Elementary Particles and Nuclear Physics (continued)

Problem 2. (continued)

b) Consider a straight section of the accelerator with length L. Estimate the neutrino intensity (number per unit area per unit time) at a distance d downstream from this straight section. What is its dependence on the beam energy?

c) There is a proposed 1 TeV machine design with circumference $s = 10^4$ m and beam of $N = 10^{13}$ muons. Estimate the neutrino event rate downstream of a 10 meter long straight section:

In a person standing 1 km away,

In a one kiloton water detector on the other side of the earth, 10^7 m away.

Monday, May 8, 2000 - 9:00 am - 12:00 noon

Part III. Section B. Elementary Particles and Nuclear Physics (continued)

3. The binding energy B for a nucleus of N neutrons and Z protons ($A = N + Z$) is given by the semi-empirical mass formula:

$$B(N,Z) = aA - bA^{\frac{2}{3}} - s\frac{(N-Z)^2}{A} - d\frac{Z^2}{A^{\frac{1}{3}}} - \frac{\delta}{A^{\frac{1}{2}}}$$

where the approximate values of the coefficients are $a = 16$ MeV, $b = 17$ MeV, $s = 23$ MeV, $d = 0.71$ MeV, and

$$\delta = \begin{cases} +11 \text{ MeV} & \text{for odd } Z \text{ and } N \\ 0 & \text{for odd } A \\ -11 \text{ MeV} & \text{for even } Z \text{ and } N \end{cases}$$

a) Give a qualitative explanation of each term in the semi-empirical mass formula.

b) Adding a gravitational binding energy term to the above formula introduces a negligible correction for any normal nucleus. However, gravitational binding energy is important for a neutron star. Derive the nuclear gravitational binding energy term in terms of A.

c) Now consider a neutron star to be a "neutron nucleus" and use the semi-empirical mass formula to calculate lower limits on the radius and mass of the star. Express your answer for the mass in units of a solar mass, $M_\odot = 2 \times 10^{30}$ kg. (The Chandrasekhar limit is an upper bound on the mass and is not relevant to this calculation.)