Department of Physics, Princeton University

Graduate Preliminary Examination
Part II

Friday, May 2, 2003
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Quantum Mechanics

1. A particle with spin one-half is at rest in a static magnetic field of strength $B_z = B_0$ oriented along the z-axis. The magnetic moment interaction splits the $S_z = +1/2$ from the $S_z = -1/2$ state One can manipulate the spin state of the particle by subjecting it to a time-dependent field $\vec{B}_1 = B_1(cos\phi(t), sin\phi(t), 0)$ that rotates in the $x - y$ plane at a variable frequency $\dot{\phi}(t)$. We will discuss different ways of choosing $\phi(t)$ to achieve the goal of transforming an initial $S_z = -1/2$ state into an $S_z = +1/2$ state.

   The two-component spin wave function ψ of this system evolves under a time-dependent Hamiltonian which can be written as

   $$H = \mu B_0 \sigma_z + \mu B_1(\sigma_x cos\phi(t) + \sigma_y sin\phi(t))$$

   where μ is the magnetic moment and $\sigma_{x,y,z}$ are the Pauli matrices (with $\sigma_x^2 = 1$ etc.). For general $\phi(t)$ this is hard to solve, but various special cases and approximations are helpful as we now show.

   (a) First show that the 'interaction picture' wave function

   $$\hat{\psi} = \exp(-i\phi(t)\frac{\sigma_z}{2})\psi$$

   evolves according to a simpler Hamiltonian $H_{rot}(t)$ which becomes time-independent when ϕ is linear in t.

   (b) Consider the case that $\dot{\phi} = \omega_1$ for a finite time interval $-T < t < T$ and is zero otherwise (this is not too hard to realize experimentally). H_{rot} is now time-independent and you can solve the Schrödinger equation. Find ω_1 and T such that a spin-down state is perfectly converted into a spin-up state by the $-T \to T$ time evolution (this is sometimes called a 'Π pulse'). Note that for this method to work for a collection of spins, they must all be subject to the same field $B_0\hat{z}$.

   (c) Now consider the case of a 'chirped' frequency such that $\dot{\phi}(t) = \alpha t$ for $-T < t < T$. $H_{rot}(t)$ now varies with time, but if its matrix elements vary slowly (i.e if α is small), and there is no level crossing, the adiabatic theorem should apply. This means that the system remains in the 'same' eigenstate of the instantaneous Hamiltonian for all time. Make a rough plot of the eigenenergies of the instantaneous Hamiltonian H_{rot} as a function of time for this case. Show that the lowest energy eigenstate evolves from spin down at $t = -T$ to spin up at $t = +T$ if we take $\alpha T \gg \mu B_0, \mu B_1$. Note that this method of spin flipping is insensitive to the value of B_0 and could work for a collection of spins in an inhomogeneous environment.

2. Consider the familiar problem of s-wave ($l = 0$) scattering of a particle of mass m from an attractive square-well potential of depth V_0 and radius r_0: $V = -V_0\theta(r_0 - r)$.

   (a) First, let's look at whether the potential has any s-wave bound states. Show that there is a critical potential strength V_{crit} such that for $0 < V_0 < V_{crit}$ there is *no* s-wave bound state. Put another way, show that a bound state (of zero energy) first appears at V_{crit}.

   (b) Set up an equation for determining the phase shift $\delta_0(k)$ and show that it implies that $\delta_0 \sim Ak$ as $k \to 0$. Evaluate the coefficient A as a function of V_0.

   (c) Show that A vanishes, as it should, as $V_0 \to 0$. More alarmingly, show that A blows up as the potential strength V_0 approaches V_{crit} from below!

   (d) Calculate the contribution of the s-wave phase shift to the total cross section in the limit of small k. How does the zero-energy cross-section behave as $V_0 \to V_{crit}$? Comment.

3. The total angular momentum of a valence electron in an atom of orbital angular momentum l can be either $J = l + 1/2$ or $J = l - 1/2$. The two total angular momentum states are typically split by spin-orbit interactions, leaving $2J + 1$ degenerate states of the same J, but different J_z. Upon applying a uniform magnetic field $B\hat{z}$, these magnetic substates are split by the interaction between the magnetic moment and the applied magnetic field. The perturbation Hamiltonian is

$$H_{int} = -\frac{e\hbar}{2m_e c} B(L_z + 2S_z) = -\frac{e\hbar}{2m_e c} B(J_z + S_z)$$

where the relative factor of two between L_z and S_z is due to the famous fact that the g-factor of the electron is two. The Zeeman splitting of the degenerate multiplet of total angular momentum J is thus

$$\delta E_{J,J_z} = -\frac{e\hbar B}{2m_e c} \langle J, J_z | (J_z + S_z) | J, J_z \rangle = -\frac{e\hbar B}{2m_e c} (J_z + \langle J, J_z | S_z | J, J_z \rangle)$$

To evaluate this explicitly, we need to solve the slightly nontrivial problem of evaluating the matrix elements of S_z.

(a) Consider the special case $l = 1$. Construct the $J = 3/2, 1/2$ states by angular momentum addition and evaluate $\langle J, J_z | S_z | J, J_z \rangle$ for all the states. Show that the energy splittings satisfy $\delta E_{J,J_z} = g_J J_z$ and state the two values of g_J that you have just computed.

(b) Now consider the case of general l. It is possible, but tedious, to directly verify that $\delta E = g_J^l J_z$ for the two possible J multiplets. Assuming that this is so (i.e. that it suffices to compute g_J^l in any one magnetic sublevel J_z), find the relevant g_J^l factors for the two multiplets $J = l \pm 1/2$.

(c) The above calculations are particular examples of a general result most easily proved using the Wigner-Eckart theorem. For a general vector operator $\vec{A}$ (one whose commutation relations with $\vec{J}$ are of the form $[J_i, A_j] = i\epsilon_{ijk} A_k$) the theorem says that

$$\langle J, M' | \vec{A} | J, M \rangle = \frac{\langle J | \vec{J} \cdot \vec{A} | J \rangle}{J(J+1)} \langle J, M' | \vec{J} | J, M \rangle$$

Use this theorem to rederive the g_J^l factor for $J = l + 1/2$ multiplet which you computed above.

Section B. Statistical Mechanics and Thermodynamics

1. An elastic string is found to have the following properties:

   - To stretch it to a total length x requires a force $f = \mu x - \alpha T + \beta T x$. Assume that α, β, μ are constants.
   - Its heat capacity at constant length x is proportional to temperature: $C_x = A(x)T$.

   We can use thermodynamic identities to derive from these facts a variety of other thermal properties. More specifically:

   (a) Calculate $\frac{\partial S}{\partial x}|_T$.

   (b) Show that A has to be independent of x.

   (c) Calculate $\frac{\partial S}{\partial T}|_x$ and give the general expression for entropy $S(x,T)$ assuming $S(0,0) = B$, where B is a constant.

   (d) Compute the heat capacity at zero tension $C_F = T\frac{\partial S}{\partial T}|_{f=0}$.

2. Let us model a white dwarf star as a degenerate Fermi gas of electrons, supported against gravitational collapse by the electron degeneracy pressure. For simplicity, we will assume that the star is a sphere of radius R and *uniform* mass density containing N electrons, N protons, and N neutrons for an approximate total mass of $M = 2Nm_p$.

   (a) First, assume that the electrons are not relativistic. Find their Fermi energy and show that at absolute zero, their total kinetic energy is

   $$U_k = \frac{3N(\hbar\pi)^2}{10m_e}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}}$$

   where V is the volume of the star. (Note that the total kinetic energy of the nucleons is much smaller than that of the electrons.)

   (b) The gravitational binding energy of a uniform-density sphere is

   $$U_{grav} = -\frac{3GM^2}{5R}$$

   Find the equilibrium radius for the white dwarf. Eliminate N. How does this radius depend on the mass?

   (c) If instead the electrons are highly relativistic, so that their energy and momentum are related by $\epsilon = cp$, then find the Fermi energy and show that the total kinetic energy is now

   $$U_k = \frac{3N\hbar\pi c}{4}\left(\frac{3N}{\pi V}\right)^{\frac{1}{3}}$$

   (d) Under what conditions is a highly relativistic degenerate electron star unstable against collapse? This is called the Chandrasekhar limit. A star that violates the limit will collapse into a neutron star or black hole, depending on whether neutron degeneracy pressure can hold up the star.

   Constants you may need:

   $$G/(\hbar c) = 6.707 \times 10^{-39}\ (\mathrm{GeV}/c^2)^{-2}$$

   $$1\ \mathrm{GeV}/c^2 = 1.78 \times 10^{-27}\ \mathrm{kg}$$

   $$M_\odot = 1.99 \times 10^{30}\ \mathrm{kg}$$

3. Consider a single free particle of mass m confined to a volume V. Let $Z_1(m)$ denote the quantum partition function for this system (where the partition sum is taken over the discrete energy levels of a particle of mass m in a box of volume V).

   (a) Show that $Z_1(m) \to V/\lambda^3$ with $\lambda = h/\sqrt{2\pi mkT}$ in the classical (or small $\hbar$) limit. Use this result to calculate the classical energy and heat capacity at fixed volume of the single particle system.

   (b) Identify the temperature at which this approximation breaks down.

   (c) Now consider a system consisting of two identical, non-interacting particles in the same box. Because of the effects of identical particle statistics, the classical expectation for the two-particle partition function $Z_2(m) = Z_1(m)^2$ is not quite correct. Show that the exact free boson and free fermion two-particle partition sums can in fact be expressed in a simple way in terms of the one-particle functions $Z_1(m)$ and $Z_1(m/2)$.

   (d) Using the classical approximation $Z_1(m) = V/\lambda^3$ derived in the first part of this problem, calculate the correction to the energy E and the heat capacity C due to Bose or Fermi statistics.