ON THE ENERGY LEVELS OF SYSTEMS WITH $Z > 137$

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(Received May 25, 1944)

The properties of the system of finite radius with $Z>137$ are investigated. It is shown that at a charge $Z=Z_{cr}$ the lower energy level of system becomes equal to $-mc^2$, which corresponds to the falling of electrons upon the nucleus. The dependence of Z_{cr} on radius of system is given.

As is is well known, the Dirac equation for an electron in a field of a point charge has discrete energy spectrum only as long as the charge is less than 137 (in electron units). For point charges greater than 137 the equation has no stationary solution for the lower energy level ($n=0$, $l=0$).

At $Z=137$ the energy which corresponds to this level becomes zero, as it seen from the usual formula:

$$W = mc^2 \left\{ 1 + \frac{\gamma^2}{(1+\sqrt{1-\gamma^2})^2} \right\}^{-1/2}$$

It is to be noted, however, that such a critical meaning of "137" takes place only for point charge. It seems to be of interest to investigate the behaviour of energy levels in system with a finite radius.

In this case the disappearance of discrete levels takes place not with the value of $W=0$ as in the case of a point charge. On the contrary, with the further increase of Z the energy of the discrete level increases further and reaches the value $-mc^2$ for some charge Z, say Z_{cr}. Only if Z becomes greater than Z_{cr}, the first discrete level disappears.

This behaviour of energy levels is not inherent in the electrically charged system. Snyder, Schiff and Weinberg (1) have shown that the value of the lower energy level of electron in a rectangular well also becomes equal to $-mc^2$, if the depth of the well arrives at some value V_0. Further increase of the well leads to the disappearance of this level and its transition into the continuous spectrum (*lower continuum*).

For the explanation of this phenomenon* suppose that we increase the depth of the well in the absence of electron. If the depth of the well becomes greater than V_0 in the lower continuum (continuous spectrum with energies $<-mc^2$), which is filled with electrons, there appears one unoccupied level originated from a discrete level. In this case the uncharged vacuum will have one unoccupied level, while in the absence of the well (or if the depth of the well is smaller than V_0, or if the charge in our case is smaller than Z_{cr}) the uncharged vacuum is a vacuum, *all* levels of which are occupied.

If an electron happens to be in such a well (or in the field of such a charge), it will jump on this unoccupied level, vacuum acquires a charge $-e$, and the electron, since this is the level of an unobservable vacuum, will pass into the unobservable state. Thus, the result is the disappearance of the electron and the decrease of the charge of the vacuum (or, in other words, of our system) by one electron

* This interpretation is given in the paper of Schiff, Snyder and Weinberg.

charge. It may be said that the electron falls onto a nucleus *.

In this paper we shall give the calculation of the value of Z_{cr} for a given finite radius r_0. Keeping this problem in mind, we shall, firstly, give the solution of the Dirac equation in this case.

We shall investigate in the following the simplest model with the potential

$$U = -\frac{Ze^2}{r_0} \qquad \text{for} \quad r < r_0$$

and

$$U = -\frac{Ze^2}{r} \qquad \text{for} \quad r > r_0.$$

Dirac equations for radial function (multiplied by r) of electron with angular momentum $l=0$ have the form:

$$\begin{aligned} \frac{df_1}{dr} - \frac{1}{r} f_1 &= (-\varkappa - w + u) f_2, \\ \frac{df_2}{dr} + \frac{1}{r} f_2 &= (-\varkappa + w - u) f_1, \end{aligned} \tag{1}$$

where $\varkappa$, w and u are the rest, total and potential energies respectively, measured in $\hbar c$ units.

For small $r < r_0$ it is easy to see that the solutions of equations (1) are:

$$\begin{aligned} f_1 &= a \sin kr, \\ f_2 &= a \cos\left(kr - \frac{\sin kr}{kr}\right), \end{aligned} \tag{2}$$

where $k = (w-u)^2 - \varkappa^2$ and a is a constant. Ii we choose r_0 sufficiently small we can put

$$k^2 = \frac{\gamma^2}{r_0^2}. \tag{3}$$

In this approximations

$$\left(\frac{f_2}{f_1}\right)_{r=r_0} = \operatorname{ctg} kr_0 - \frac{1}{kr_0} \approx \operatorname{ctg}\gamma - \frac{1}{\gamma}. \tag{4}$$

We denote

$$\gamma = \frac{Z^2 e^2}{\hbar c}$$

and

$$\gamma \operatorname{ctg}\gamma - 1 = \zeta(\gamma) \tag{5}$$

then

$$\left(\frac{f_2}{f_1}\right)_{r=r_0} = \gamma\zeta(\gamma). \tag{6}$$

For $r > r_0$, but, in general, small, the solution of (1) has the form:

$$\begin{aligned} f_1 &= b\,(r^{ig} + \beta r^{-ig}), \\ f_2 &= \frac{b}{\gamma}\,[(1-ig)\,r^{ig} + \beta\,(1+ig)\,r^{-ig}], \end{aligned} \tag{7}$$

where b and β are constants, the later can be determined from the conditions of continuity for $r = r_0$ * and $g^2 = \gamma^2 - 1$.

We have from (7):

$$\left(\frac{f_2}{f_1}\right)_{r=r_0} = \gamma\,\frac{1 + \beta r_0^{-2ig}}{(1-ig) + (1+ig)\,\beta r^{-2ig}}, \tag{8}$$

comparing (6) and (8) we obtain:

$$\beta = -\frac{\zeta(\gamma)(1-ig) - 1}{\zeta(\gamma)(1+ig) - 1}\, r_0^{2ig}. \tag{9}$$

Thus, for $r > r_0$ (near the origin) the wave functions are:

$$\begin{aligned} f_1 &= A\left\{[1 - \zeta(\gamma)(1+ig)]\left(\frac{r}{r_0}\right)^{ig} - \text{compl. conj.}\right\}, \\ f_2 &= A\left\{\frac{1-ig}{\gamma}[1 - \zeta(\gamma)(1+ig)]\left(\frac{r}{r_0}\right)^{ig} - \text{compl. conj.}\right\}. \end{aligned} \tag{10}$$

For obtaining the equation for the energy levels we must investigate the behaviour of the solution at infinity.

For this purpose we shall bring the equations in a canonical form of the equations for Whittaker functions (2).

* In the case of an integer spin a "fall" of this sort is connected with the complex eigenvalues. The Dirac equation cannot have complex eigenvalues, and in this case the "fall" will be of such a peculiar character.

* Normalization constants a and b have no importance for us, and we shall not determine them later.

Instead of functions f_1 and f_2, we introduce new functions G_1 and G_2 defined by the equations ([3]):

$$G_1 = f_1 \sin\frac{\varepsilon}{2} + f_2 \cos\frac{\varepsilon}{2}, \qquad G_2 = -f_1 \sin\frac{\varepsilon}{2} + f_2 \cos\frac{\varepsilon}{2}, \tag{11}$$

where $\varepsilon = \arccos W/mc^2$ (W is the energy).

We introduce now the new variable

$$x = 2\left(1 - \frac{W}{mc^2}\right)^{1/2} \frac{mc}{\hbar} r_0. \tag{12}$$

For this function we obtain the following equations:

$$\frac{dG_1}{dx} + \frac{1}{x} G_2 = -\frac{1}{2} G_1 + \frac{\gamma}{x \sin\varepsilon}(G_1 \cos\varepsilon - G_2), \qquad \frac{dG_2}{dx} + \frac{1}{x} G_1 = \frac{1}{2} G_2 + \frac{\gamma}{x \sin\varepsilon}(G_1 - G_2 \cos\varepsilon). \tag{13}$$

Eliminating from this equations G_1 and G_2 correspondingly, we arrive at the equations of the second order:

$$\frac{d^2G}{dx^2} + \frac{1}{x}\frac{dG}{dx} + \left[-\frac{1}{4} + \frac{\gamma \operatorname{ctg}\varepsilon \pm \frac{1}{2}}{x} - \frac{1-\gamma^2}{x^2}\right] G = 0, \tag{14}$$

where the double sign ($\pm$) in the second term in the square bracket corresponds to the two equations for G_1 and G_2 respectively.

If we put now

$$G = x^{-1/2} G' \tag{15}$$

we arrive at the Whittaker equations for G':

$$\frac{d^2G'}{dx^2} + \left[-\frac{1}{4} + \frac{\gamma \operatorname{ctg}\varepsilon \pm \frac{1}{2}}{x} + \frac{\frac{1}{4} - (1-\gamma^2)}{x^2}\right] G' = 0. \tag{16}$$

As it is shown in ([2]), the solutions of (16) are Whittaker functions:

$$G' = W_{ig,\, \gamma \operatorname{ctg}\varepsilon \pm 1/2}(x). \tag{17}$$

We have chosen the solution which disappears at infinity as $e^{-x/2}$. The second solution of (16) increases at infinity as $e^{x/2}$ and does not satisfy the conditions of our problem.

Using the expansion of Whittaker function we obtain from (16) and (17), finally (for small x):

$$G_1 = c\left\{\frac{\Gamma(-2ig)}{\Gamma(-ig - \gamma \operatorname{ctg}\varepsilon)} x^{ig} + \text{compl. conj.}\right\},$$
$$G_2 = c\left(-1 + \frac{\gamma}{\sin\varepsilon}\right)\left\{\frac{\Gamma(-2ig)}{\Gamma(-ig - \gamma \operatorname{ctg}\varepsilon)} \frac{x^{ig}}{ig + \gamma \operatorname{ctg}} + \text{compl. conj.}\right\}, \tag{18}$$

c is a new normalization constant.

It is possible also to construct the function G_1 and G_2 from the solutions (10) by means of equations (11). We obtain immediately:

$$G_1 = A \sin\frac{\varepsilon}{2}\left\{\left(1 + \frac{1-ig}{\gamma} \operatorname{ctg}\frac{\varepsilon}{2}\right)[1 - \zeta(\gamma)(1+ig)]\left(\frac{r}{r_0}\right)^{ig} - \text{compl. conj.}\right\}. \tag{19}$$

and a similar expression for G_2*.

It is obvious that apart from normalization constants the functions (18) and (19) must be identical. This is just the condition we have looked for, which determines the energy levels.

As G_2 can be obtained from G_1 by means of (13), it is sufficient to compare the expressions for G_1 only.

Noting the different signs in curly brackets of (18) and (19) the following equations can be obtained:

$$\operatorname{Re} \frac{[1 - \zeta(\gamma)(1 - ig)]\left(1 + \frac{1+ig}{\gamma} \operatorname{ctg}\frac{\varepsilon}{2}\right)\Gamma(-2ig)\, x_0^{ig}}{\Gamma(-ig - \gamma \operatorname{ctg}\varepsilon)} = 0. \tag{20}$$

* In the substitution of the solutions (18) and (19) in (13) the following relations are useful:

$$\frac{+\gamma + (1-ig)\operatorname{ctg}\frac{\varepsilon}{2}}{-\gamma + (1-ig)\operatorname{ctg}\frac{\varepsilon}{2}} = \frac{1 + \gamma/\sin\varepsilon}{-ig + \gamma \operatorname{ctg}\varepsilon} = \frac{ig - \gamma \operatorname{ctg}\varepsilon}{\gamma/\sin\varepsilon - 1}.$$

where according to (12):

$$x_0 = 2\left(1-\frac{W}{mc^2}\right)^{1/2}\frac{mc}{\hbar}r_0. \qquad (21)$$

This equation determines for the given value of r_0, the values of energy levels of our system.

For our problem there are essential the energy levels close to $-mc^2$ only. In this case the equation (21) may be simplified. Since $\operatorname{ctg}\varepsilon$ is very great (and negative) we can use the asymptotic expansion of Γ-function (Stirling's formula), then

$$\arg\Gamma(-ig-\gamma\operatorname{ctg}\varepsilon) = -g\lg(-\gamma\operatorname{ctg}\varepsilon) = -g\lg\frac{\gamma W/mc^2}{(1-W/mc^2)^{1/2}}.$$

Computing the argument of the expression under the Re sign in (20) putting $\operatorname{ctg}\varepsilon/2=0$ (since if $W\approx-\mu c^2$ $\varepsilon\approx\pi$) and using the obvious relation

$$\arg\Gamma(-2ig) = \frac{\pi}{2} + \arg\Gamma(1-2ig)$$

we obtain:

$$g\lg\gamma\frac{2mcr_0}{\hbar}\frac{|W|}{mc^2} + \arg\Gamma(1-2ig) + \arg\operatorname{tg}\frac{g\zeta(\gamma)}{1-\zeta(\gamma)} = n\pi \qquad (22)$$
$$(n=1,\ 2,\ \ldots).$$

It must be noted that from (20) follows the double sign (±) before $n\pi$, but for a given r_0 the different signs correspond to g which differ also only in sign. As the sign of g is not determined (only g^2 enters the problem) we retain only the minus sign alone and consider g as a positive quantity.

The value $n=0$ must be excluded, since in this case we obtain discrete negative energy level for $n=0$ $(Z=137)$, which tends to $-\infty$ if $r\to0$. But it can be shown that exact equations have no discrete negative levels even for a point charge.

Putting now in (22) $|W|=mc^2$ and $u=0$, we obtain the equation for the value of a critical charge for a given "radius" of a nucleus r_0, *i. e.* the minimum charge in the field of which the first energy level passes into a lower continuum:

$$\lg\gamma_{cr}\frac{2mcr_0}{\hbar} = \frac{1}{g_{cr}}\left[-\operatorname{arc\,tg}\frac{g_{cr}\zeta(\gamma_{cr})}{1-\zeta(\gamma_{cr})} - \arg\Gamma(1-2ig_{cr}) - \pi\right]. \qquad (23)$$

For a very small r_0 $(\ll\hbar/2\,mc)$, equation (23) may be written in approximate form:

$$g_{cr} = -\frac{\pi}{\lg\frac{2mcr_0}{\hbar}}. \qquad (24)$$

Thus for a nucleus of finite radius the disappearance of discrete levels (and falling of electrons) happens not for $Z=137$, but for a greater charge, determined by equation (23).

For example, for a value $r_0=1.2\cdot10^{-12}$, $Z_{cr}=200$, for $r_0=8.10^{-13}$, $Z_{cr}=175$.

In conclusion we wish to thank Prof. L. Landau for interesting discussions in connection with this problem.

Translated by authors.

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