

# Density-Matrix Description of the EPR “Paradox”

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## 1 Problem

A qbit version of the Einstein-Podolsky-Rosen “paradox” [1] involves a 2-qbit entangled state,

$$\frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}, \quad (1)$$

that is observed/measured some time after its creation by Alice and Bob, who are separated by a distance large compared to the spatial extents of the two qbits (labeled A and B). If Alice and Bob measure the qbits in the same basis, and later compared their results, these will be either  $|0\rangle_A|0\rangle_B$  or  $|1\rangle_A|1\rangle_B$ , with 50% probability each if the experiment is repeated many times.

This correlation led Einstein and others to suppose that either the initial state was actually a mixture of the two states  $|0\rangle_A|0\rangle_B$  and  $|1\rangle_A|1\rangle_B$  rather than the superposition (1), or that the observation of qbit A by Alice somehow caused qbit B to take on a value correlated with that found by Alice (or that the observation of qbit B by Bob somehow caused qbit A to take on a value correlated with that found by Bob).

Experiments that evaluate certain inequalities, first suggested by Bohm and Aharonov [2], and later a stronger version due to Bell [3], confirm that the initial, entangled state is not a mixture.<sup>1</sup> These impressive results leave many people with the uncomfortable feeling that quantum phenomena either involve faster-than-light “influences,” or that Nature is “nonlocal.”

Discussions of the Einstein-Podolsky-Rosen “paradox” are usually given in terms of wavefunctions, which are not the technical language appropriate if mixtures of states are involved. Give a discussion using density matrix operators (as introduced independently(?) in 1927 by Landau and by von Neumann).

## 2 Solution

A review of density-matrix/operator formalism is given in the Appendix.

As discussed in Appendix A.4, the density-matrix operator for the entangled state (1) is

$$\rho_{AB} \left( \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \right) = \frac{|00\rangle \pm |11\rangle \langle 00| \pm \langle 11|}{\sqrt{2}} = \frac{|00\rangle\langle 00| \pm |00\rangle\langle 11| \pm |11\rangle\langle 00| + |11\rangle\langle 11|}{2}. \quad (2)$$

However, once the entangled qbits A and B become spatially separated, Alice can no longer be sure that they are well described by eq. (2). Her ignorance of how Bob may or

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<sup>1</sup>A survey of these experiments up to 1981 is given in [4].

may not have acted upon qbit B can be formally represented by taking the trace of operator (2) over B, resulting in the reduced density operator,

$$\begin{aligned} \rho_A &= \text{tr}_B(\rho_{AB}) = \frac{|0\rangle\langle 0| \langle 0|0\rangle \pm |0\rangle\langle 1| \langle 0|1\rangle \pm |1\rangle\langle 0| \langle 1|0\rangle + |1\rangle\langle 1| \langle 1|1\rangle}{2} \\ &= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbf{I}}{2}. \end{aligned} \quad (3)$$

Although the original system was prepared as a pure state, Alice’s knowledge of that system (when she is out of lightspeed communication with Bob and the second qbit) is as if her qbit was prepared as a 50:50 mixed state of  $|0\rangle_A$  and  $|1\rangle_A$ . Nothing Bob does changes her understanding of the qbit A (unless/until Bob sends her information about his activities via a lightspeed communications channel), and when she measures it, she finds it to be a  $|0\rangle$  with 50% probability, or a  $|1\rangle$  with 50% probability, independent of the history of second qbit.<sup>2</sup> Likewise nothing Alice does changes Bob’s understanding of the qbit B (unless/until Alice sends him information about her activities via a lightspeed communications channel).

If Alice observes qbit A to be  $|0\rangle_A$ , she can then characterize her understanding of the system as  $|0\rangle_A|0\rangle_B$ , but this does not mean that she knows that qbit B has been or will be measured. Rather, it only means that if she somehow knows that qbit B will be measured by Bob (in the  $|0\rangle$ - $|1\rangle$  basis), she can predict that the result will be  $|0\rangle_B$ . If qbits A and B are spacelike separated, Bob cannot know of this “prediction” of Alice prior to his observation of qbit B.

Does qbit B have the “value”  $|0\rangle_B$  simply because Alice predicts this to be so after her observation that qbit A has the value  $|0\rangle_A$ , as implied by Einstein, Podolsky and Rosen [1]? The answer to this is NO. As noted by Peres [7], “Unperformed experiments have no results.” That is, after qbits A and B are prepared in the entangled state (1), these qbits do not have definite “values” until each of them has been measured, say by Alice and Bob.

Neither Alice nor Bob can predict the values of the qbit that they measure, and they also consider that this qbit did not have a definite “value” prior to that measurement, although later Alice may learn that Bob made a correct prediction as to what value Alice would measure (and *vice versa*).

Alice and Bob should consider that their measurements of the qbit that reaches each of them permits them to predict the result of a measurement by the other, but their own measurement did not give a “value” to the other qbit.

The density matrix/operator formalism codifies how the knowledge of Alice and Bob about the system of two entangled qbits is incomplete in different ways at different times, but this does not imply that the formalism itself (*i.e.*, quantum theory) is “incomplete.”

Einstein often said that the “real” situations of two spatially separated (sub)systems should be independent of one another (meaning that one system cannot affect the other in a way that implies faster-than-light transmission of a signal).

If separate observers A and B (who are at rest with respect to each other) look at their respective systems at the same time (as determined by previously synchronized clocks), the

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<sup>2</sup>Skeptics might infer that the procedure  $\rho_A = \text{tr}_B(\rho_{AB})$  to represent Alice’s knowledge of the entangled state (1) is incorrect. For further justification of its validity, see Box 2.6, p. 107 of Nielsen and Chuang [6]

combined results of their observations of the spatially encoded qbit will be that exactly one of A or B finds a particle present in their system.

Einstein appears to have concluded from this that the initial state of the system AB was not the pure state (1), but rather was the mixed state,

$$\rho_{\text{Einstein}} = \frac{|0\rangle_A \langle 0|_A |0\rangle_B \langle 0|_B}{2} + \frac{|1\rangle_A \langle 1|_A |1\rangle_B \langle 1|_B}{2}. \quad (4)$$

An interpretation of eq. (4) is that the qbits “really” were  $|0\rangle_A|0\rangle_B$  or  $|1\rangle_A|1\rangle_B$  all along, but we didn’t know which until we “looked.”

Note that the formal knowledge of observer A, if ignorant about system B (as seems natural if subsystems A and B are spatially separated), is the same for both the pure state (2) and for Einstein’s mixed state (4),

$$\rho_A = \text{tr}_B(\rho_{AB}) = \text{tr}_B(\rho_{\text{Einstein}}) = \frac{\mathbf{I}}{2}. \quad (5)$$

I would like to argue (with Einstein, if he were alive), that this shows how the characterization of knowledge of quantum systems via density operators satisfies the criterion of separability that Einstein insisted upon.

However, it remains that the pure state (1)/(2) is a more subtle entity than Einstein’s mixed state (4). We claim that pure states such as eqs. (1)/(2) can exist in Nature, and that the qbits “really” do not have definite values until they both are measured. Furthermore, the entangled state includes a correlation between the two qbit whose consequences are simple to described by whose description of a “real” property of the system does not fit well with a “classical” views.

The author concludes from the debate on the Einstein-Podolsky-Rosen “paradox” that it shows the “classical” view to be “incomplete” in that it cannot accommodate the concept of entanglement, while the existence of this concept in the quantum view is not evidence that this latter view is “incomplete.”

## A Appendix: Density Matrices

### A.1 Wave Function of a Pure State

If a quantum system is in an idealized pure state, we have characterized this by a wave function

$$|\psi\rangle = \sum_j \psi_j |j\rangle, \quad (6)$$

that is a weighted sum of basis states  $|j\rangle$ . The time evolution of state  $|\psi\rangle$  has been described by a unitary transformation  $\mathbf{U}(t, t')$  such that

$$|\psi(t')\rangle = \mathbf{U}(t, t')|\psi(t)\rangle. \quad (7)$$

If the state  $|\psi\rangle$  is observed via a (hermitian) measurement operator  $\mathbf{M}$  whose eigenvectors are the basis states  $|j\rangle$  with corresponding eigenvalues  $m_j$ , then we can write

$$\mathbf{M} = \sum_j \mathbf{M}_j = \sum_j m_j \cdot \mathbf{P}_j = \sum_j m_j \cdot |j\rangle\langle j|, \quad (8)$$

and the probability that the result of the measurement is that state  $|\psi\rangle$  is found in basis state  $|j\rangle$  is

$$P_j = \langle\psi|\mathbf{P}_j^\dagger\mathbf{P}_j|\psi\rangle = \langle\psi|\mathbf{P}_j|\psi\rangle = \langle\psi|j\rangle\langle j|\psi\rangle = |\langle j|\psi\rangle|^2. \quad (9)$$

The probable value (or expectation value) of variable  $m$  for state  $|\psi\rangle$  is thus

$$\langle m \rangle = \sum_j m_j P_j = \sum_j m_j \langle\psi|\mathbf{P}_j|\psi\rangle = \langle\psi|\sum_j m_j \cdot \mathbf{P}_j|\psi\rangle = \langle\psi|\mathbf{M}|\psi\rangle. \quad (10)$$

## A.2 Density Matrix of a Pure State

The density operator  $\rho$  of a pure state (6) is simply its corresponding projection operator,

$$\rho = |\psi\rangle\langle\psi| = \sum_{j,k} \psi_j \psi_k^* |j\rangle\langle k| \quad (\text{pure state}). \quad (11)$$

Clearly, the operator  $\rho$  can be represented by the hermitian matrix whose elements are

$$\rho_{jk} = \psi_j \psi_k^*. \quad (12)$$

Examples:

$$\rho(|0\rangle) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho(|1\rangle) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(|\pm\rangle) = \rho\left(\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}\right) = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \quad (13)$$

$$\rho(|00\rangle) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho\left(\frac{|00\rangle \pm |11\rangle}{\sqrt{2}}\right) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}, \quad (14)$$

$$\rho\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \rho\left(\frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}\right) = \frac{1}{4} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}. \quad (15)$$

Basis states have only a single nonzero (diagonal) element to their density matrices. A pure state that is a superposition of basis states has nonzero off-diagonal elements to its density matrix.

Other properties of density matrices follow immediately: The square of the density operator of a pure state is itself,

$$\rho^2 = \rho \quad (\text{pure state}). \quad (16)$$

The trace of the density matrix is 1,

$$\text{tr}(\boldsymbol{\rho}) = \sum_j \boldsymbol{\rho}_{jj} = \sum_j \psi_j \psi_j^* = \sum_j |\psi_j|^2 = 1, \quad (17)$$

since quantum states are normalized to unit probability. An alternative derivation of this is

$$\text{tr}(\boldsymbol{\rho}) = \text{tr}(|\psi\rangle\langle\psi|) = \sum_j \langle j|\psi\rangle\langle\psi|j\rangle = \sum_j \langle\psi|j\rangle\langle j|\psi\rangle = \langle\psi|\psi\rangle = 1. \quad (18)$$

The time evolution of the density operator follows from (7) as

$$\boldsymbol{\rho}(t') = |\psi(t')\rangle\langle\psi(t')| = \mathbf{U}(t, t')|\psi(t)\rangle\langle\psi(t)|\mathbf{U}^\dagger(t, t') = \mathbf{U}(t, t')\boldsymbol{\rho}(t)\mathbf{U}^\dagger(t, t'). \quad (19)$$

The probability that state  $|\psi\rangle$  is found in basis state  $|j\rangle$  as the result of a measurement follows from (9) as

$$\begin{aligned} P_j &= |\langle j|\psi\rangle|^2 = \langle\psi|j\rangle\langle j|\psi\rangle = \langle j|\psi\rangle\langle\psi|j\rangle = \langle j|j\rangle\langle j|\psi\rangle\langle\psi|j\rangle = \sum_k \langle k|j\rangle\langle j|\psi\rangle\langle\psi|k\rangle \\ &= \text{tr}(|j\rangle\langle j|\psi\rangle\langle\psi|) = \text{tr}(\mathbf{P}_j\boldsymbol{\rho}), \end{aligned} \quad (20)$$

which is a special case of the general result for an operator  $\mathbf{O}$  that

$$\langle\psi|\mathbf{O}|\psi\rangle = \text{tr}(\mathbf{O}\boldsymbol{\rho}). \quad (21)$$

The expectation value  $\langle m \rangle$  for a measurement of state  $|\psi\rangle$  using operator  $\mathbf{M} = \sum_j m_j \cdot \mathbf{P}_j = \sum_j m_j \cdot |j\rangle\langle j|$  follows from (21) as

$$\langle m \rangle = \langle\psi|\mathbf{M}|\psi\rangle = \sum_j m_j \langle\psi|\mathbf{P}_j|\psi\rangle = \sum_j m_j \text{tr}(\mathbf{P}_j\boldsymbol{\rho}) = \text{tr}(\mathbf{M}\boldsymbol{\rho}). \quad (22)$$

### A.3 Density Matrix of a Mixed State

These results show that the density-matrix description of a pure quantum state recovers all the features of the usual description. However, there does not yet appear to be any advantage to the use of density matrices. That advantage lies in the ease with which the density-matrix description can be extended to include so-called **mixed states** in which the quantum state is one of a set of pure states  $|\psi_i\rangle$  with probability  $P_i$ , where the total probability is, of course, unity:  $\sum_i P_i = 1$ . In this case, we define

$$\boldsymbol{\rho} = \sum_i P_i |\psi_i\rangle\langle\psi_i| = \sum_i P_i \boldsymbol{\rho}_i \quad (\text{mixed state}). \quad (23)$$

We readily set that the mixed-state density matrix (23) obeys all of the properties (17)-(22). However,

$$\boldsymbol{\rho}^2 \neq \boldsymbol{\rho} \quad (\text{mixed state}), \quad (24)$$

which provides a means of determining whether a given density matrix describes a pure state or a mixed state.

Example: A 50:50 mixture of states  $|00\rangle$  and  $|11\rangle$  has density matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (25)$$

A mixture of basis states has no off-diagonal elements in its density matrix.

Example: A 50:50 mixture of states  $|0\rangle$  and  $|1\rangle$  has the same density matrix as a 50:50 mixture of states  $|+\rangle$  and  $|-\rangle$ . From eq. (13) we have,

$$\rho = \frac{\rho(|0\rangle) + \rho(|1\rangle)}{2} = \frac{\rho(|+\rangle) + \rho(|-\rangle)}{2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbf{I}}{2}. \quad (26)$$

Indeed, a rotation  $R_y(\phi)$  of basis states  $|0\rangle$  and  $|1\rangle$  by angle  $\theta$  about the  $y$ -axis in Bloch space<sup>3</sup> leads to the new basis states  $\cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle$  and  $\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle$ . Hence, a 50:50 mixture of these new basis states also has density matrix (26).

A historic debate about the meaning of the quantum wave functions concerns whether they reflect that Nature is intrinsically probabilistic or that the probabilities merely reflect our ignorance of some underlying well-defined “reality.” We argue that a pure state is one for which probabilities are intrinsic.<sup>4</sup> In contrast, a mixed state (23) can be regarded as actually being in one of its component pure states  $|\psi_i\rangle$ , but we don’t know which.<sup>5</sup> The probabilities  $P_i$  in eq. (23) summarize our ignorance/knowledge of which pure states are present, while the coefficients  $\psi_j$  in eq. (6) represent intrinsic probabilities (strictly, probability amplitudes) as to what can be observed of the pure state  $|\psi\rangle$ .

Mixed states and their density-matrix description are therefore useful in quantum statistical mechanics in which we are ignorant of details of the state of our system or ensemble of systems.

## A.4 Density Matrix of a Composite System

The density-matrix description is also useful when dealing with a system for which we have different qualities of information about its component subsystems.

Consider a system with two subsystems A and B for which the density matrix of the whole system is  $\rho_{AB}$ . If our knowledge of system B is limited, we may wish to consider what we can say about system A only. That is, we desire the density matrix  $\rho_A$ .

The claim is that the appropriate procedure is to calculate

$$\rho_A = \text{tr}_B(\rho_{AB}), \quad (27)$$

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<sup>3</sup>See, for examples, prob. 4 of [5].

<sup>4</sup>For a recent review of the Kochen-Specker theorem [8] that a quantum-mechanical spin-1/2 state cannot “really” have simultaneous definite values of its spin vector along three orthogonal axes, see [9].

<sup>5</sup>The example of eq. (26) reminds us that a given mixed-state density matrix corresponds to different mixtures in different bases, so considerable quantum subtlety remains even for mixed states.

where the trace over subsystem B can be accomplished with the aid of the definition

$$\begin{aligned}\mathrm{tr}_B(|A_1B_1\rangle\langle A_2B_2|) &= \mathrm{tr}_B(|A_1\rangle\langle A_2| \otimes |B_1\rangle\langle B_2|) = |A_1\rangle\langle A_2| \mathrm{tr}_B(|B_1\rangle\langle B_2|) \\ &= |A_1\rangle\langle A_2| \langle B_1|B_2\rangle,\end{aligned}\tag{28}$$

recalling eq. (18).

If subsystems A and B have no nontrivial couplings, then  $\rho_{AB} = \rho_A \otimes \rho_B$ , so that  $\mathrm{tr}(\rho_{AB}) = \mathrm{tr}(\rho_A \otimes \rho_B) = \rho_A \mathrm{tr}(\rho_B) = \rho_A$ , as expected.

Of greater interest is the case when subsystems A and B are entangled. For example, consider the entangled 2-qbits states described by the righthand case in eq. (14). To apply eq. (28) it is easier to rewrite the density matrix (14) as a density operator,

$$\rho_{AB} \left( \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \right) = \frac{|00\rangle \pm |11\rangle \langle 00| \pm \langle 11|}{\sqrt{2}} = \frac{|00\rangle\langle 00| \pm |00\rangle\langle 11| \pm |11\rangle\langle 00| + |11\rangle\langle 11|}{2}.\tag{29}$$

Then,

$$\begin{aligned}\rho_A &= \mathrm{tr}_B(\rho_{AB}) = \frac{|0\rangle\langle 0| \langle 0|0\rangle \pm |0\rangle\langle 1| \langle 0|1\rangle \pm |1\rangle\langle 0| \langle 1|0\rangle + |1\rangle\langle 1| \langle 1|1\rangle}{2} \\ &= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbf{I}}{2}.\end{aligned}\tag{30}$$

This noteworthy result helps us understand why Alice could not extract any information about Bob's observations of the second qbit of an entangled, but spatially separated 2-qbit system, as considered in prob. 6(b) of [5], by her observation of the first qbit. Since the two qbits are spatially separated, Alice is ignorant of the state of Bob's qbit, and her density-matrix description of the system is obtained by taking the trace over the second qbit. Although the original system was prepared as a pure state, Alice's knowledge of that system is as if her qbit was prepared as a 50:50 mixed state of  $|0\rangle_A$  and  $|1\rangle_A$ . Nothing Bob does changes Alice's understanding of the first qbit, and when she measures it, she finds it to be a  $|0\rangle$  with 50% probability, or a  $|1\rangle$  with 50% probability, independent of the history of second qbit.

Skeptics, however, might infer from the result (30) that the claim (27) is incorrect. For further justification of its validity, see Box 2.6, p. 107 of [6].<sup>6</sup>

## A.5 Density Matrix of a Qbit

Show that the density matrix for a qbit can be written as

$$\rho = \frac{\mathbf{I} + \mathbf{r} \cdot \boldsymbol{\sigma}}{2},\tag{31}$$

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<sup>6</sup>The density-matrix explanation of why we cannot expect controversial results from observation of one of two entangled subsystems could have been given by Bohr [10] as an answer to the EPR "paradox" [1] in 1935, but it was not. Some enthusiasts of EPR's argument obliquely acknowledge the impact of the density matrix by referring to it as the "destiny matrix".

where  $\mathbf{r}$  is a real 3-vector with  $|\mathbf{r}| \leq 1$ , and the maximum holds only if the qbit is in a pure state. What is the unit vector  $\hat{\mathbf{r}}$  that corresponds to the pure state

$$|\psi\rangle = e^{i\gamma} \left[ \cos \frac{\alpha}{2} |0\rangle + e^{i\beta} \sin \frac{\alpha}{2} |1\rangle \right]? \quad (32)$$

The density matrix for a qbit is a  $2 \times 2$  hermitian matrix whose trace is 1. So we can write

$$\boldsymbol{\rho} = \frac{\mathbf{I} + \mathbf{A}}{2}, \quad (33)$$

where matrix  $\mathbf{A}$  is also hermitian, but with zero trace. Then, we have

$$\mathbf{A} = \begin{pmatrix} z & b \\ c & -z \end{pmatrix} = \mathbf{A}^\dagger = \begin{pmatrix} z^* & c^* \\ b^* & -z^* \end{pmatrix}. \quad (34)$$

Therefore,  $z$  is real, and if  $b = x - iy$  then  $c = x + iy$ , so that

$$\mathbf{A} = x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{r} \cdot \boldsymbol{\sigma}, \quad (35)$$

where  $\mathbf{r} = (x, y, z)$  is a real 3-vector. Thus

$$\boldsymbol{\rho} = \frac{\mathbf{I} + \mathbf{r} \cdot \boldsymbol{\sigma}}{2}, \quad (31)$$

as claimed.

If this density matrix represents a pure state, then

$$\frac{\mathbf{I} + \mathbf{r} \cdot \boldsymbol{\sigma}}{2} = \boldsymbol{\rho} = \boldsymbol{\rho}^2 = \frac{\mathbf{I} + 2 \mathbf{r} \cdot \boldsymbol{\sigma} + (\mathbf{r} \cdot \boldsymbol{\sigma})^2}{4} = \frac{\mathbf{I}(1 + |\mathbf{r}|^2) + 2 \mathbf{r} \cdot \boldsymbol{\sigma}}{4}, \quad (36)$$

recalling that for any two ordinary vectors  $\mathbf{a}$  and  $\mathbf{b}$ ,

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{I} + i \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}. \quad (37)$$

Hence,  $|\mathbf{r}|^2 = 1$  for a pure state.

If the density matrix represents a mixed state,

$$\boldsymbol{\rho} = \sum_i P_i |\psi_i\rangle \langle \psi_i| = \sum_i P_i \boldsymbol{\rho}_i, \quad (23)$$

then each of the component pure-state density matrices can be represented in the form (31) with a corresponding unit vector  $\hat{\mathbf{r}}_i$ . Thus, the vector  $\mathbf{r}$  for the density matrix (23) obeys

$$\mathbf{r} = \sum_i P_i \hat{\mathbf{r}}_i, \quad \text{where} \quad \sum_i P_i = 1. \quad (38)$$

Hence  $|\mathbf{r}| \leq 1$ , and the bound is achieved only if all  $\hat{\mathbf{r}}_i$  are identical, in which case the density matrix actually represents a pure state.



For a pure state

$$|\psi\rangle = e^{i\gamma} \left[ \cos \frac{\alpha}{2} |0\rangle + e^{i\beta} \sin \frac{\alpha}{2} |1\rangle \right], \quad (32)$$

the density matrix is

$$\begin{aligned} \rho &= \begin{pmatrix} \cos^2 \frac{\alpha}{2} & \cos \alpha \sin \alpha e^{-i\beta} \\ \cos \alpha \sin \alpha e^{i\beta} & \sin^2 \frac{\alpha}{2} \end{pmatrix} \\ &= \frac{\mathbf{I} + \begin{pmatrix} \cos \alpha & \sin \alpha (\cos \beta - i \sin \beta) \\ \sin \alpha (\cos \beta + i \sin \beta) & -\cos \alpha \end{pmatrix}}{2}. \end{aligned} \quad (39)$$

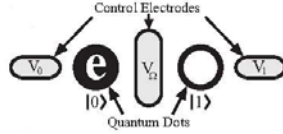
From this we read off the components of  $\hat{\mathbf{r}}$  as

$$\hat{\mathbf{r}} = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha), \quad (40)$$

which corresponds to a unit vector in the direction  $(\alpha, \beta)$  in a spherical coordinate system in Bloch space, consistent with the geometric interpretation of a qbit (as in prob. 4 of [5]).

## A.6 Density Matrix of a Pair of Quantum Dots

One type of spatially encoded qbit consists of a pair of quantum dots [= regions in a thin silicon layer where electrodes define a potential minimum that can “trap” electrons (Earnshaw’s theorem applies in three dimensions, but not in two)]; the states  $|0\rangle$  and  $|1\rangle$  are defined by the presence of an electron on one or the other of the two quantum dots, as sketched in the figure below (from [11]).



We can also think of a single spatially encoded qbit as consisting of a pair of qbits. Suppose that state  $|0\rangle$  ( $|1\rangle$ ) corresponds to the presence of a particle in region A (B). Then we can consider the first qbit to have the two states  $|0\rangle_A$  and  $|\text{vac}\rangle_A$ , where the “vacuum” state occurs when there is no particle in region A. Similarly, the second qbit consists of the two states  $|1\rangle_B$  and  $|\text{vac}\rangle_B$ . That is,

$$|0\rangle = |0\rangle_A |\text{vac}\rangle_B, \quad \text{and} \quad |1\rangle = |\text{vac}\rangle_A |1\rangle_B, \quad (41)$$

The system AB also supports the 2-bit states  $|\text{vac}\rangle_A |\text{vac}\rangle_B$  and  $|0\rangle_A |1\rangle_B$ , but these are not to be used for our spatially encoded qbit.

The density operator  $\rho$  for the pure states  $|\pm\rangle = (|0\rangle_A |\text{vac}\rangle_B \pm |\text{vac}\rangle_A |1\rangle_B) / \sqrt{2}$  is

$$\begin{aligned} \rho &= |\pm\rangle\langle\pm| = \frac{|0\rangle_A |\text{vac}\rangle_B \pm |\text{vac}\rangle_A |1\rangle_B}{\sqrt{2}} \frac{\langle 0|_A \langle \text{vac}|_B \pm \langle \text{vac}|_A \langle 1|_B}{\sqrt{2}} \\ &= \frac{|0\rangle_A \langle 0|_A |\text{vac}\rangle_B \langle \text{vac}|_B}{2} + \frac{|\text{vac}\rangle_A \langle \text{vac}|_A |1\rangle_B \langle 1|_B}{2} \\ &\quad \pm \frac{|0\rangle_A \langle \text{vac}|_A |\text{vac}\rangle_B \langle 1|_B}{2} \pm \frac{|\text{vac}\rangle_A \langle 0|_A |1\rangle_B \langle \text{vac}|_B}{2}. \end{aligned} \quad (42)$$

Taking the partial trace over subsystem B with the aid of eq. (28), we find

$$\begin{aligned}\rho_A &= \text{tr}_B(\rho) = \frac{|0\rangle_A \langle 0|_A \langle \text{vac}_B | \text{vac}_B \rangle}{2} + \frac{|\text{vac}\rangle_A \langle \text{vac}|_A \langle 1_B | 1_B \rangle}{2} \\ &\quad \pm \frac{|0\rangle_A \langle \text{vac}|_A \langle \text{vac}_B | 1_B \rangle}{2} + \frac{|\text{vac}\rangle_A \langle 0|_A \langle 1_B | \text{vac}_B \rangle}{2} \\ &= \frac{|0\rangle_A \langle 0|_A \langle \text{vac}_B | \text{vac}_B \rangle}{2} + \frac{|\text{vac}\rangle_A \langle \text{vac}|_A \langle 1_B | 1_B \rangle}{2} = \frac{\mathbf{I}}{2}.\end{aligned}\tag{43}$$

The pure state (42) of our spatially encoded qbit is an entangled state of the qbits of subsystems A and B. However, from the perspective of an observer of subsystem A who is ignorant of subsystem B, the state of A is given by eq. (43) which is a mixed state comprised of  $|0\rangle_A$  with 50% probability, and state  $|\text{vac}\rangle_A$  with 50% probability.

The observation of a particle in region A allows the observer of that region to conclude that the particle is not in region B, but this conclusion does not affect region B or a potential observer thereof, who stills thinks that region might contain the particle if (s)he has not yet looked at it. If the second observer does not actually observe/measure region B, (s)he cannot say that there is no particle there.

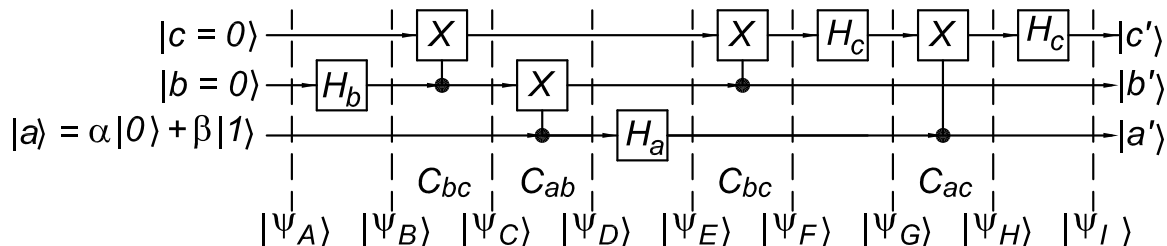
That is, the discussion of the pair of quantum dots as an entangled state (42) of two qbits is essentially identical to the main discussion of this problem.

## A.7 Density Matrices for Quantum Teleportation

Recall that in the circuit for quantum teleportation [12, 13], discussed in prob. 6(d) of [5], Alice makes her measurements of bits  $|a\rangle$  and  $|b\rangle$  when the wave function of the system of three qbits  $|a\rangle$ ,  $|b\rangle$  and  $|c\rangle$  is

$$|\psi_E\rangle = \alpha \frac{|000\rangle + |100\rangle + |011\rangle + |111\rangle}{2} + \beta \frac{|010\rangle - |110\rangle + |001\rangle - |101\rangle}{2},\tag{44}$$

where the initial state of the qbit  $a$  was  $|a\rangle = \alpha|0\rangle + \beta|1\rangle$ . The teleportation scheme is represented in the sketch below, in which time flows from left to right.



We consider the reduced density matrix of the system at the intermediate time when  $|\psi\rangle = |\psi_E\rangle$  from Bob's point of view. Recall that Bob has only bit  $|c\rangle$  at this time, so his knowledge of the system is described by tracing over bits  $|a\rangle$  and  $|b\rangle$  in the full density matrix. Note that at this time Bob does not have knowledge of the initial state of bit  $|a\rangle$ , and must await receipt of Alice's results of her measurements of bits  $|a\rangle$  and  $|b\rangle$  of  $|\psi_E\rangle$  before he can reconstruct the initial state of  $|a\rangle$ .

To take the partial trace over bits  $|a\rangle$  and  $|b\rangle$ , the appropriate version of the rule (28) is

$$\text{tr}_{AB}(|A_1 B_1 C_1\rangle \langle A_2 B_2 C_2|) = |C_1\rangle \langle C_2| \langle A_1 B_1 | A_2 B_2\rangle.\tag{45}$$

Since the bits  $|0\rangle_c$  and  $|1\rangle_c$  each appear in two of the basis states of eq. (44) that are multiplied by  $\alpha$  and in two that are multiplied by  $\beta$ , the Bob's reduced density matrix simplifies to

$$\rho_{\text{Bob,E}} = \text{tr}_{ab}(|\psi_E\rangle\langle\psi_E|) = \frac{|\alpha|^2 + |\beta|^2}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{\mathbf{I}}{2}. \quad (46)$$

So, at the time when the state of the system is  $|\psi_E\rangle$ , Bob's knowledge of the system, as summarized in eq. (46) includes no information about the amplitudes  $\alpha$  and  $\beta$  of the initial state of bit  $|a\rangle$ .

Only after getting additional information from Alice (via communication at lightspeed or less) can he convert his density matrix [at step  $I$  of the figure of the solution to prob. 6(d) of [5]] to

$$\rho_{\text{Bob,I}} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} = (\alpha|0\rangle + \beta|1\rangle)(\alpha^*\langle 0| + \beta^*\langle 1|). \quad (47)$$

For completeness, we note that the density matrix that represents Alice's knowledge only, when the system is in state  $|\psi\rangle_E$ , is

$$\rho_{\text{Alice,E}} = \text{tr}_c(\rho_E) = \frac{1}{4} \begin{pmatrix} 1 & 2\text{Re}(\alpha\beta^*) & |\alpha|^2 - |\beta|^2 & 2\text{Re}(\alpha\beta^*) \\ 2\text{Re}(\alpha\beta^*) & 1 & 2i\text{Im}(\alpha\beta^*) & |\alpha|^2 - |\beta|^2 \\ |\alpha|^2 - |\beta|^2 & -2i\text{Im}(\alpha\beta^*) & 1 & -2\text{Re}(\alpha\beta^*) \\ 2\text{Re}(\alpha\beta^*) & |\alpha|^2 - |\beta|^2 & -2\text{Re}(\alpha\beta^*) & 1 \end{pmatrix}, \quad (48)$$

which has nontrivial off-diagonal elements that contain information as to the initial state  $|a\rangle = \alpha|0\rangle + \beta|1\rangle$ .

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