

Does Destructive Interference Destroy Energy?

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1 Problem

In principle, a pair of counterpropagating waves (with separate sources) whose waveforms are the negative of each other can completely cancel at some moment in time. Does this destructive interference also destroy the energy of the waves at this moment?

Comment also on the case of waves from a single source.

2 Solution

While the answer is well-known to be **NO**, and energy is conserved in the superposition of waves, discussion of this is sparse in textbooks.¹

Also, it appears that the trivial case of a null wave (with zero energy) is often mistakenly described as an example of destructive interference of two waves moving with opposite amplitudes (but the same energies) in the same direction, which has led to the misimpression that destructive interference can destroy energy.

A one-dimensional wave moving in one direction can have only one source, and there can be only one such wave at a given point, such that wave interference is not a relevant concept here. We can write $0 = \sin(kx - \omega t) - \sin(kx - \omega t)$, but this mathematical identity does not have the physical implication that two distinct waves are present, each with nonzero energy.

The concept of wave interference applies only to waves from different sources, or to the propagation in two or more dimensions of a wave from a single source.

When two sources are present the waves cannot propagate only in the same direction; such waves must be counterpropagating if they are one dimensional, or they propagate in two or more dimensions. The key here is that the energy associated with a wave has two forms, generically called “kinetic” and “potential,” that are equal for a wave propagating in a single direction, while in the case of destructive (or constructive) interference of counterpropagating waves one form of energy decreases and the other increases such that the total energy remains constant. The character of energy conservation in the case of two sources is well illustrated by a pair of counterpropagating waves in one dimension, as discussed in sec. 2.1.

¹Thanks to Hans Schantz for pointing this out. Pedagogic discussions of this issue include R.C. Levine, *False paradoxes in superposition of electric and acoustic waves*, Am. J. Phys. **48**, 28 (1980), http://physics.princeton.edu/~mcdonald/examples/EM/levine_ajp_48_28_80.pdf
W.N. Mathews, Jr, *Superposition and energy conservation for small amplitude mechanical waves*, Am. J. Phys. **54**, 233 (1986), http://physics.princeton.edu/~mcdonald/examples/mechanics/mathews_ajp_54_233_86.pdf
N. Gauthier, *What happens to energy and momentum when two oppositely-moving wave pulses overlap?* Am. J. Phys. **71**, 787 (2003), http://physics.princeton.edu/~mcdonald/examples/EM/gauthier_ajp_71_787_03.pdf
R. Drosd, L. Minkin and A.S. Shapovalov, *Interference and the Law of Energy Conservation*, Phys. Teach. **52**, 428 (2014), http://physics.princeton.edu/~mcdonald/examples/mechanics/drosd_pt_52_428_14.pdf

Interference in a wave from a single source can only occur for propagation in two or more dimensions where the system includes entities that “scatter” one portion of the wave onto another portion. The key here is that energy which would appear in one region in the absence of “scattering” appears elsewhere in its presence. The character of energy conservation in the case of a single source is illustrated by double-slit interference in sec. 2.2.

2.1 Counterpropagating Waves from Two Sources in One Dimension

Here, we present three related arguments for counterpropagating one-dimensional waves.

2.1.1 Transverse Waves on a Stretched String

A string of linear mass density ρ under tension T has wave speed

$$c = \sqrt{\frac{T}{\rho}}. \quad (1)$$

Writing the transverse displacement as $y(x, t)$, the kinetic energy associated with this waveform is

$$\text{KE} = \int \frac{\rho \dot{y}^2}{2} dx, \quad (2)$$

where $\dot{y} = dy/dt$. The potential energy can be taken the work done in stretching the string,

$$\text{PE} = \int T dl = \int T \left(\sqrt{1 + y'^2} - 1 \right) dx \approx \int \frac{T y'^2}{2} dx, \quad (3)$$

where $y' = dy/dx$. The first derivatives for traveling waves $y(x \pm ct)$ are related by

$$\dot{y} = \pm c y', \quad \dot{y}^2 = c^2 y'^2 = \frac{T}{\rho} y'^2, \quad (4)$$

which implies that $\text{KE} = \text{PE}$ for a wave traveling in a single direction, and that the total energy U of such a wave is given by

$$U = \text{KE} + \text{PE} = 2\text{KE} = 2\text{PE}. \quad (5)$$

For two waves propagating in opposite directions with similar waveforms,

$$y_1(x, t) = y(x - ct), \quad y_2(x, t) = \pm y_1(x, -t) = \pm y(x + ct). \quad (6)$$

The total energy of the waves when they don't overlap is

$$U_{\text{total}} = \text{KE}_1 + \text{PE}_1 + \text{KE}_2 + \text{PE}_2 + 2\text{KE} + 2\text{PE} = 4\text{KE} = 4\text{PE}, \quad (7)$$

where the energies without subscripts are the common values associated with the individual waves. The first derivatives for the superposition $y_{\text{tot}} = y_1 + y_2$ of the two waves are

$$\dot{y}_{\text{tot}} = \dot{y}_1 + \dot{y}_2, = -c [y'(x - ct) \mp y'(x + ct)] \quad y'_{\text{tot}} = y'_1 + y'_2 = y'(x - ct) \pm y'(x + ct), \quad (8)$$

Destructive Interference at $t = 0$

Destructive interference corresponds to the lower signs in eq. (8), in which case we have at time $t = 0$,

$$\dot{y}_{\text{tot}}(t = 0) = 2\dot{y}(x), \quad y'_{\text{tot}}(t = 0) = 0, \quad (9)$$

$$\text{KE}_{\text{tot}}(t = 0) = \int \frac{\rho \dot{y}_{\text{tot}}^2}{2} dx = 4 \int \frac{\rho \dot{y}^2}{2} dx = 4\text{KE}, \quad (10)$$

$$\text{PE}_{\text{tot}}(t = 0) \approx \int \frac{\text{T}y'_{\text{tot}}{}^2}{2} dx = 0, \quad (11)$$

$$U_{\text{tot}}(t = 0) = \text{KE}_{\text{tot}}(t = 0) + \text{PE}_{\text{tot}}(t = 0) = 4\text{KE} = U_{\text{total}}, \quad (12)$$

such that energy is conserved. Destructive interference destroys the potential energy, but doubles the kinetic energy.

Constructive Interference at $t = 0$

Similarly, constructive interference corresponds to the upper signs in eq. (8), in which case we have at time $t = 0$,

$$\dot{y}_{\text{tot}}(t = 0) = 0, \quad y'_{\text{tot}}(t = 0) = 2y'(x), \quad (13)$$

$$\text{KE}_{\text{tot}}(t = 0) = \int \frac{\rho \dot{y}_{\text{tot}}^2}{2} dx = 0, \quad (14)$$

$$\text{PE}_{\text{tot}}(t = 0) \approx \int \frac{\text{T}y'_{\text{tot}}{}^2}{2} dx = 4 \int \frac{\text{T}y'^2}{2} dx = 4\text{PE}, \quad (15)$$

$$U_{\text{tot}}(t = 0) = \text{KE}_{\text{tot}}(t = 0) + \text{PE}_{\text{tot}}(t = 0) = 4\text{PE} = U_{\text{total}}. \quad (16)$$

and again energy is conserved. Constructive interference doubles the potential energy, but destroys the kinetic energy.

2.1.2 Transmission Line

A two-conductor transmission line is characterized by capacitance C and inductance L per unit length (ignoring the electrical resistance of the conductors). We consider the voltage difference $V(x, t)$ between the two conductors, which carry equal and opposite currents $\pm I(x, t)$. Then, the speed of waves along the line is

$$c = \frac{1}{\sqrt{LC}}, \quad (17)$$

and the voltage and currents are related by

$$V = \pm IZ, \quad Z = \sqrt{\frac{L}{C}}, \quad (18)$$

where the upper(lower) sign holds for waves moving in the $+x(-x)$ direction, and Z is the (real) transmission-line impedance.²

²See, for example, K.T. McDonald, *Distortionless Transmission Line* (Nov. 11, 1996), <http://physics.princeton.edu/~mcdonald/examples/distortionless.pdf>.

The energy of a wave has both capacitive and inductive terms,

$$U_C = \int \frac{CV(x)^2}{2} dx, \quad U_L = \int \frac{LI(x)^2}{2} dx, \quad (19)$$

For a wave that moves only in a single direction,

$$U_C = \int \frac{CV(x)^2}{2} dx = \int \frac{CZ^2I(x)^2}{2} dx = \int \frac{LI(x)^2}{2} dx = U_L, \quad (20)$$

and the total energy of such a wave is

$$U = U_C + U_L = 2U_C = 2U_L. \quad (21)$$

For two waves propagating in opposite directions with similar waveforms,

$$V_1(x, t) = V(x - ct), \quad V_2(x, t) = \pm V_1(x, -t) = \pm V(x + ct), \quad (22)$$

$$I_1(x, t) = \frac{V_1}{Z} = \frac{V(x - ct)}{Z} = I(x - ct), \quad I_2(x, t) = -\frac{V_2}{Z} = \mp I(x + vt). \quad (23)$$

The total energy of the waves when they don't overlap is

$$U_{\text{total}} = U_{C,1} + U_{L,1} + U_{C,2} + U_{L,2} = 2U_C + 2U_L = 4U_C = 4U_L, \quad (24)$$

where the energies without subscripts are the common values associated with the individual waves. The total voltage and current for the superposition of the two waves are

$$V_{\text{tot}} = V_1 + V_2 = V(x - ct) \pm V(x + ct), \quad I_{\text{tot}} = I_1 + I_2 = I(x - ct) \mp I(x + ct). \quad (25)$$

Destructive Interference at $t = 0$

Destructive interference corresponds to the lower signs in eq. (25), in which case we have at time $t = 0$,

$$V_{\text{tot}}(t = 0) = 2V(x), \quad I_{\text{tot}}(t = 0) = 0, \quad (26)$$

$$U_{C,\text{tot}}(t = 0) = \int \frac{CV_{\text{tot}}^2}{2} dx = 4 \int \frac{CV^2}{2} dx = 4U_C, \quad (27)$$

$$U_{L,\text{tot}}(t = 0) = \int \frac{LI_{\text{tot}}^2}{2} dx = 0, \quad (28)$$

$$U_{\text{tot}}(t = 0) = U_{C,\text{tot}}(t = 0) + U_{L,\text{tot}}(t = 0) = 4U_C = U_{\text{total}}, \quad (29)$$

such that energy is conserved. Destructive interference destroys the inductive energy, but doubles the capacitive energy.

Constructive Interference at $t = 0$

Similarly, constructive interference corresponds to the upper signs in eq. (25), in which case we have at time $t = 0$,

$$V_{\text{tot}}(t = 0) = 0, \quad I_{\text{tot}}(t = 0) = 2I(x), \quad (30)$$

$$U_{C,\text{tot}}(t = 0) = \int \frac{CV_{\text{tot}}^2}{2} dx = 0, \quad (31)$$

$$U_{L,\text{tot}}(t = 0) = \int \frac{LI_{\text{tot}}^2}{2} dx = 4 \int \frac{LI^2}{2} dx = 4U_L, \quad (32)$$

$$U_{\text{tot}}(t = 0) = U_{C,\text{tot}}(t = 0) + U_{L,\text{tot}}(t = 0) = 4U_L = U_{\text{total}}, \quad (33)$$

and again energy is conserved. Constructive interference doubles the inductive energy, but destroys the capacitive energy.

2.1.3 Plane Electromagnetic Waves

A plane electromagnetic wave propagating in vacuum in the x -direction with, say, y polarization had electric and magnetic fields (in Gaussian units)

$$\mathbf{E} = \mathbf{E}(x - ct) = E(x - ct) \hat{\mathbf{y}}, \quad \mathbf{B} = \mathbf{B}(x - ct) = E(x - ct) \hat{\mathbf{z}}, \quad (34)$$

where c is the speed of light in vacuum.

The energy of a wave has both electric and magnetic terms,

$$U_E = \int \frac{E^2}{8\pi} d\text{Vol}, \quad U_M = \int \frac{B^2}{8\pi} d\text{Vol} = U_E, \quad (35)$$

and the total energy of such a wave is

$$U = U_E + U_M = 2U_E = 2U_M. \quad (36)$$

For two waves propagating in opposite directions with similar waveforms,

$$\mathbf{E}_1(x, t) = \mathbf{E}(x - ct), \quad \mathbf{B}_1(x, t) = \mathbf{B}(x - ct), \quad (37)$$

$$\mathbf{E}_2(x, t) = \pm \mathbf{E}(x + ct), \quad \mathbf{B}_2(x, t) = \mp \mathbf{B}(x + ct). \quad (38)$$

The total energy of the waves when they don't overlap is

$$U_{\text{total}} = U_{E,1} + U_{M,1} + U_{E,2} + U_{M,2} = 2U_E + 2U_M = 4U_E = 4U_M, \quad (39)$$

where the energies without subscripts are the common values associated with the individual waves. The total electric and magnetic fields for the superposition of the two waves are

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}(x - ct) \pm \mathbf{E}(x + ct), \quad \mathbf{B}_{\text{tot}} = \mathbf{B}_1 + \mathbf{B}_2 = \mathbf{B}(x - ct) \mp \mathbf{B}(x + ct). \quad (40)$$

Destructive Interference at $t = 0$

Destructive interference corresponds to the lower signs in eq. (40), in which case we have at time $t = 0$,

$$\mathbf{E}_{\text{tot}}(t = 0) = 2\mathbf{E}(x), \quad \mathbf{B}_{\text{tot}}(t = 0) = 0, \quad (41)$$

$$U_{E,\text{tot}}(t = 0) = \int \frac{E_{\text{tot}}^2}{8\pi} d\text{Vol} = 4 \int \frac{E^2}{8\pi} d\text{Vol} = 4U_E, \quad (42)$$

$$U_{M,\text{tot}}(t = 0) = \int \frac{B_{\text{tot}}^2}{8\pi} d\text{Vol} = 0, \quad (43)$$

$$U_{\text{tot}}(t = 0) = U_{E,\text{tot}}(t = 0) + U_{B,\text{tot}}(t = 0) = 4U_E = U_{\text{total}}, \quad (44)$$

such that energy is conserved. Destructive interference destroys the magnetic energy, but doubles the electric energy.

Constructive Interference at $t = 0$

Similarly, constructive interference corresponds to the upper signs in eq. (40), in which case we have at time $t = 0$,

$$\mathbf{E}_{\text{tot}}(t = 0) = 0, \quad \mathbf{B}_{\text{tot}}(t = 0) = 2\mathbf{B}(x), \quad (45)$$

$$U_{E,\text{tot}}(t = 0) = \int \frac{E_{\text{tot}}^2}{8\pi} d\text{Vol} = 0, \quad (46)$$

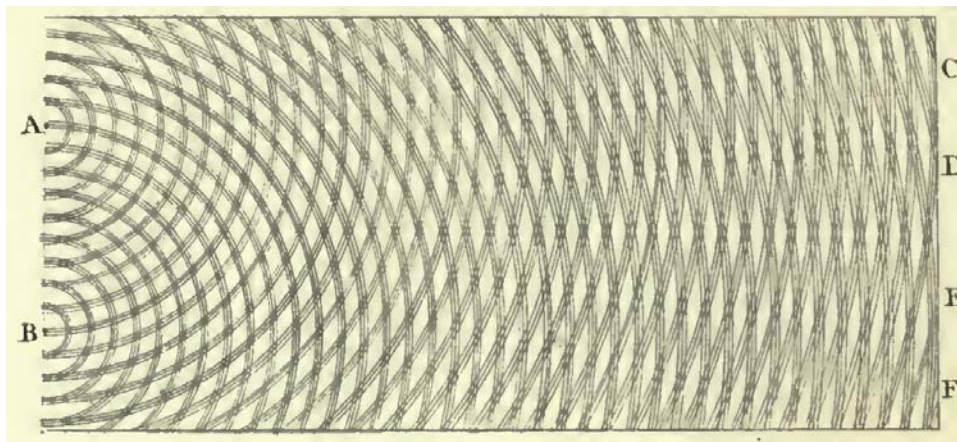
$$U_{M,\text{tot}}(t = 0) = \int \frac{B_{\text{tot}}^2}{8\pi} d\text{Vol} = 4 \int \frac{B^2}{2} d\text{Vol} = 4U_M, \quad (47)$$

$$U_{\text{tot}}(t = 0) = U_{E,\text{tot}}(t = 0) + U_{M,\text{tot}}(t = 0) = 4U_M = U_{\text{total}}, \quad (48)$$

and again energy is conserved. Constructive interference doubles the magnetic energy, but destroys the electric energy.

2.2 The Double Slit Experiment with a Single Source

Following Young,³ we consider a line source of cylindrical light waves of wavelength λ that impinge on a planar screen that is parallel to the line source at closest distance large compared to λ . The screen has two narrow line slits, parallel to the line source and separated by distance D , located symmetrically about the line on the screen closest to the source. Beyond the planar screen is a cylindrical screen of radius $R \gg \lambda$ on which the intensity of the light is observed.



We first to consider scalar waves (*i.e.*, scalar diffraction theory), with amplitude $A e^{-i\omega t}$ at each of the two slits, where A is a complex number, $\omega = kc = 2\pi c/\lambda$ and c is the speed of light in vacuum. The intensity of a wave at some point is proportional to the absolute square of its amplitude there. We suppose the units of A are such that the (time-average) power per unit length passing through each slit is A^2 . The intensity on the cylindrical screen is uniform, with total power $P_0 = A^2$ per unit length, and hence the angular distribution of (time-average) power per unit length on the half-cylinder screen from the light that passes

³T. Young, *On the Nature of Light and Colours*, Lecture 39, *Course of Lectures on Natural Philosophy and Mechanical Arts* (London, 1897),

http://physics.princeton.edu/~mcdonald/examples/optics/young_lecture_39.pdf.

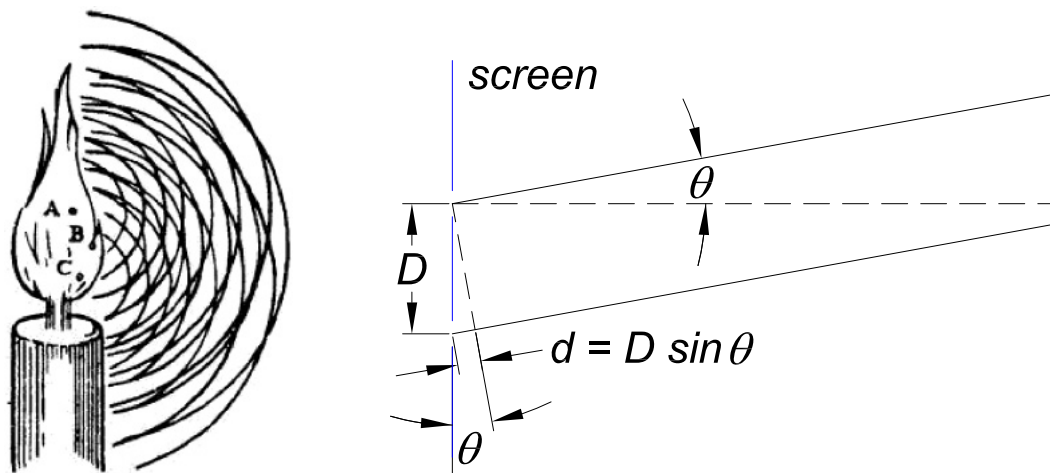
through a single slit is

$$\frac{dP}{d\theta} = \frac{P_0}{\pi} = \frac{A^2}{\pi}, \quad P = \int_{-\pi/2}^{\pi/2} \frac{dP}{d\theta} d\theta = P_0. \quad (49)$$

In the absence of interference, the power incident on the distance cylindrical screen would be $2P_0$.

The interference pattern of the waves beyond the slits can be computed according to the method of Huygens.⁴ The path length for light traveling at angle θ from the two slits to the distant screen has path difference $d = D \sin \theta$, and hence phase difference

$$\Delta\phi = \frac{2\pi d}{\lambda} = \frac{2\pi D \sin \theta}{\lambda} = kD \sin \theta. \quad (50)$$



The (time-average) power incident on the distant cylindrical screen at angle θ depends on the absolute square of the sum of the amplitudes of the light that passes through the two slits, and has angular distribution $dP/d\theta$ and total power P given by

$$\frac{dP}{d\theta} = \frac{P_0 |1 + e^{i\Delta\phi}|^2}{\pi} = \frac{2P_0}{\pi} (1 + \cos \Delta\phi), \quad (51)$$

$$P = \int_{-\pi/2}^{\pi/2} \frac{dP}{d\theta} d\theta = \frac{2P_0}{\pi} \int_{-\pi/2}^{\pi/2} [1 + \cos(kD \sin \theta)] d\theta = 2P_0 [1 + J_0(kD)], \quad (52)$$

noting that⁵

$$J_0(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(x \sin \theta) d\theta. \quad (53)$$

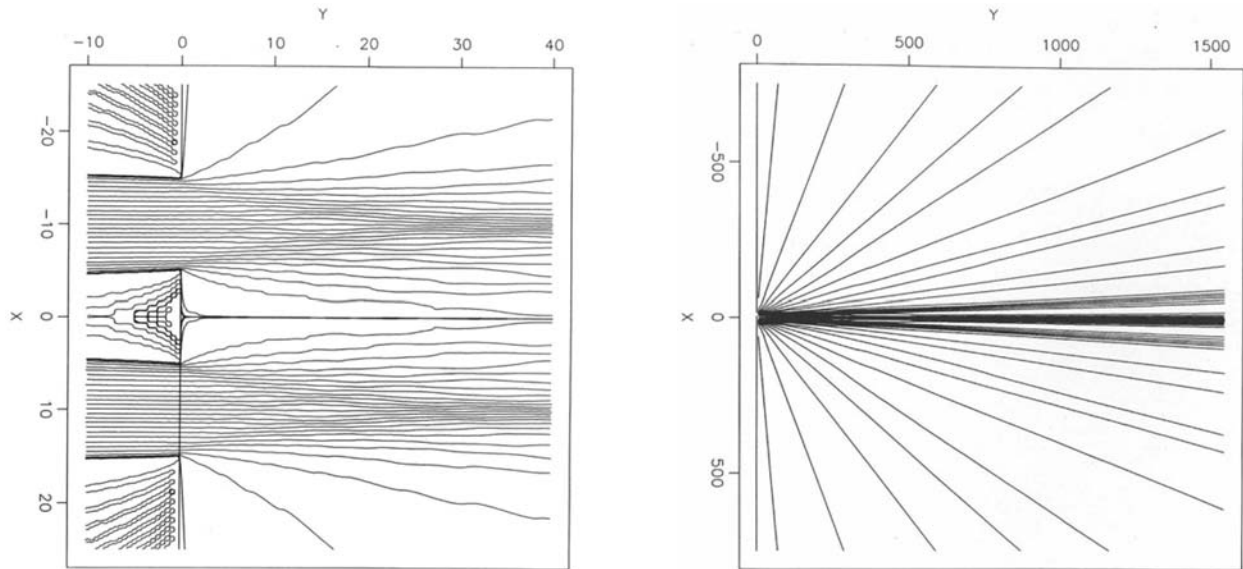
Since the Bessel function $J_0(x)$ oscillates about zero, energy is conserved only “on average” in an analysis of the double-slit experiment via scalar diffraction theory.⁶

⁴C. Huygens, *Treatise on Light* (1678, English translation, Macmillan, 1912), p. 21, http://physics.princeton.edu/~mcdonald/examples/optics/huygens_treatise_on_light.pdf.

⁵See, for example, eq. 9.1.18 of M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (NBS, 1964), http://physics.princeton.edu/~mcdonald/examples/EM/abramowitz_and_stegun.pdf.

⁶This is noted in sec. 5 of A. Lundin, *Interference and Energy Conservation in Phased Antenna Arrays and Young’s Double Slit Experiment* (Dec. 12, 2012), <http://www.diva-portal.org/smash/get/diva2:576304/FULLTEXT01.pdf>.

A more accurate theory of electromagnetic waves characterizes the flow of energy via the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.^{7,8} An analytic solution for the double-slit experiment apparently does not exist, but numerical computations are reported by Jeffers *et al.*,⁹ with lines of (energy-conserving) Poynting flux as shown below. The electromagnetic analysis matches that of scalar diffraction theory for small angles to the direction of incidence, but differs at large angles such that energy is conserved in detail for any slit separation.



The essence of energy conservation in waves from a single source is that regions of higher energy density are compensated by regions of lower energy density.¹⁰

⁷J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884), http://physics.princeton.edu/~mcdonald/examples/EM/Poynting_ptrsl175_343_84.pdf.

⁸In so-called Bohmian quantum mechanics a photon is a point particle whose “Bohmian trajectory” is essentially a line of the Poynting vector field. See, for example, C. Philippidis, C. Dewdney and B.J. Hiley, *Quantum Interference and the Quantum Potential*, Nuovo Cim. **52B**, 15 (1979), http://physics.princeton.edu/~mcdonald/examples/QM/philippidis_nc_52b_15_79.pdf.

⁹S. Jeffers *et al.*, *Classical Electromagnetic Theory of Diffraction and Interference: Edge, Single-Slit and Double-Slit Solutions*, in *Waves and Particles in Light and Matter*, A. van der Merwe and A. Garuccio, eds. (Plenum Press, New York, 1994), p. 309, http://physics.princeton.edu/~mcdonald/examples/EM/jeffers_wplm_309_94.pdf.

¹⁰Comments on energy conservation in the closely related case of 3-dimensional waves from two sources are given, for example, in Y.-S. Hoh, *On the electromagnetic wave omnidirectional interference phenomena*, Am. J. Phys. **55**, 570 (1987), http://physics.princeton.edu/~mcdonald/examples/EM/hoh_ajp_55_570_87.pdf.