

Four Expressions for Electromagnetic Field Momentum

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1 Problem

Show that electromagnetic field momentum can be written for static fields in four equivalent forms,

$$\mathbf{p}_{\text{EM}} = \int \frac{\rho \mathbf{A}}{c} d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \int \frac{V \mathbf{J}}{c^2} d\text{Vol} = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} d\text{Vol}, \quad (1)$$

in Gaussian units, where ρ is the (total) electric charge density, $\mathbf{A} = \int \mathbf{J} d\text{Vol}/cr$ is the magnetic vector potential (where $\nabla \cdot \mathbf{A} = 0$ for static fields in both the Coulomb and Lorenz gauges), \mathbf{J} is the (total) electric current density, c is the speed of light in vacuum, $\mathbf{E} = \int \rho \hat{\mathbf{r}} d\text{Vol}/r^2$ is the electric field, $\mathbf{B} = \int \mathbf{J} \times \hat{\mathbf{r}} d\text{Vol}/cr^2$ is the magnetic field, and $V = \int \rho d\text{Vol}/r$ is the electric scalar potential.¹

The first form of eq. (1) is due to Maxwell (sec. 57 of [2]), who regarded the vector potential \mathbf{A} at the location of an electric charge q as providing a measure, $q\mathbf{A}/c$, of electromagnetic momentum and an interpretation of Faraday's electrotonic state (secs. 60-61 of [3]). That Faraday associated with some kind of momentum with this state is hinted in sec. 1077 of [4]. The second form is due to J.J. Thomson, who speculated as to electromagnetic mass/momentum (1881) [5], and continued with a concept of momentum stored in the electromagnetic field (1891) [6, 7], with the field-momentum density being the Poynting vector [8] divided by c^2 , $\mathbf{p}_{\text{EM}} = \mathbf{S}/c^2$, where c is the speed of light in vacuum.² The third form was introduced by Furry [10],³ and the fourth form is due to Aharonov *et al.* [13].

Of these four forms, the second depends only on the electromagnetic fields, and so is the most "Maxwellian". As such, we consider the volume density of electromagnetic field momentum to be⁴

$$\mathbf{p}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}, \quad (2)$$

and not $\rho \mathbf{A}/c$, $V \mathbf{J}/c^2$ or $(\mathbf{J} \cdot \mathbf{E}) \mathbf{r}/c^2$.

Also discuss which of the four forms of eq. (1) lead to valid expressions for electromagnetic-field angular momentum.⁵

¹For discussion of alternative forms of electromagnetic energy, momentum and angular momentum for fields with arbitrary time dependence, see, for example [1].

²This relation was also advocated by Poincaré [9].

³The density $V \mathbf{J}/c^2$ is the static limit of an expression for the field-momentum density proposed by Livens (1926), bottom of p. 263 of [11]. This density is also the static limit of \mathbf{S}/c^2 for several of the alternative forms of the Poynting vector proposed by Slepian (1942) [12].

⁴We avoid here the famous Abraham-Minkowski debate as to the form of electromagnetic field momentum in media. Some comments by the author on this topic are at [14, 15].

⁵Electromagnetic field angular momentum was first computed for a special case by Darboux [16] and by Poincaré [17], without their realizing the physical significance of the computations [18]. The first explicit mention of electromagnetic field momentum appears to be by J.J. Thomson in [19]. See also [20].

2 Solution

2.1 Minkowski's Derivation of the Poynting-Poincaré Form

We follow an argument of Minkowski [21] as to field momentum by considering the total force density on electromagnetic media (of unit permittivity and unit permeability),

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = \mathbf{f} = \rho\mathbf{E} + \frac{\mathbf{J}}{c} \times \mathbf{B}, \quad (3)$$

where \mathbf{p}_{mech} is the density of mechanical momentum in the media.⁶ Using the Maxwell equations $\nabla \cdot \mathbf{E} = 4\pi\rho$ and $\nabla \times \mathbf{B} = 4\pi\mathbf{J}/c + (1/c)\partial\mathbf{E}/\partial t$ for the macroscopic fields,⁷

$$\begin{aligned} \frac{d\mathbf{p}_{\text{mech}}}{dt} &= \frac{\mathbf{E}(\nabla \cdot \mathbf{E})}{4\pi} - \mathbf{B} \times \frac{1}{4\pi} \left(\nabla \times \mathbf{B} + \mathbf{B} \times \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= -\frac{\partial}{\partial t} \left(\frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \right) + \frac{1}{4\pi} [\mathbf{E}(\nabla \cdot \mathbf{E}) + \mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}) - \mathbf{B} \times (\nabla \times \mathbf{B})] \\ &\equiv -\frac{\partial \mathbf{p}_{\text{EM}}}{\partial t} + \nabla \cdot \mathbb{T}_{\text{EM}}, \end{aligned} \quad (4)$$

where

$$\mathbf{p}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} = \frac{\mathbf{S}}{c^2}, \quad (5)$$

is the density of momentum associated with the electromagnetic field,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \quad (6)$$

is the Poynting vector, and

$$\mathbb{T}_{\text{EM},ij} = \frac{1}{4\pi} \left[E_i E_j + B_i B_j - \delta_{ij} \frac{E^2 + B^2}{2} \right] \quad (7)$$

is the symmetric Maxwell stress 3-tensor associated with the electromagnetic fields.⁸ To arrive at eq. (7) we note that,

$$\begin{aligned} [\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E})]_i &= E_i \frac{\partial E_j}{\partial x_j} - E_j \frac{\partial E_j}{\partial x_i} + E_j \frac{\partial E_i}{\partial x_j} \\ &= \frac{\partial}{\partial x_j} \left[E_i E_j - \delta_{ij} \frac{\mathbf{E} \cdot \mathbf{E}}{2} \right]. \end{aligned} \quad (8)$$

The total electromagnetic field momentum (in the form associated with Poynting, Thomson and Poincaré) follows from eq. (5) as

$$\mathbf{P}_{\text{EM}_P} = \int \mathbf{p}_{\text{EM}} d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}. \quad (9)$$

⁶Subtle difficulties with the Lorentz force density (3) for permeable media are considered in [22].

⁷Minkowski [21] actually used ρ_{free} and \mathbf{J}_{free} rather than the total charge and current densities ρ and \mathbf{J} , as well as the auxiliary fields \mathbf{D} and \mathbf{H} .

⁸For an extension of this argument if magnetic monopoles existed, see [23].

2.2 Equivalence of the Faraday-Maxwell and Poynting-Poincaré Forms

This section is based on an argument by Vladimir Hnizdo. The result that $\mathbf{P}_{\text{EM}_P} = \mathbf{P}_{\text{EM}_M}$ was first deduced for a special case by Thomson [19], who gave the first definition of electromagnetic field angular momentum, \mathbf{L}_{EM_P} , in the same paper (which is also the origin of the Feynman cylinder paradox [24]). The results of this section may have first been given, via a more compact derivation, by Trammel [25], and independently by Calkin [26] for linear momentum only.⁹

Static electromagnetic fields \mathbf{E} and \mathbf{B} can be characterized by the time-independent Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{E} = 0, \quad (10)$$

where ρ is the electric charge density, and

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \quad (11)$$

which implies that the time-independent current density \mathbf{J} satisfies $\nabla \cdot \mathbf{J} = 0$.

For present purposes we can avoid use of the current density \mathbf{J} and instead consider the vector potential \mathbf{A} , which has zero divergence in the Coulomb gauge (and also in the Lorentz gauge for static problems),

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla \cdot \mathbf{A} = 0. \quad (12)$$

To confirm that the electromagnetic momentum

$$\mathbf{P}_{\text{EM}_M} = \int \frac{\rho\mathbf{A}}{c} d\text{Vol} \quad (13)$$

is equal to

$$\mathbf{P}_{\text{EM}_P} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}, \quad (14)$$

and that the electromagnetic angular momentum

$$\mathbf{L}_{\text{EM}_M} = \int \mathbf{r} \times \mathbf{P}_{\text{EM}_M} d\text{Vol} = \int \mathbf{r} \times \frac{e\mathbf{A}}{c} d\text{Vol} \quad (15)$$

is equal to

$$\mathbf{L}_{\text{EM}_P} = \int \frac{\mathbf{r} \times (\mathbf{E} \times \mathbf{B})}{4\pi c} d\text{Vol}, \quad (16)$$

we show that $\mathbf{E} \times \mathbf{B}$ is equal to $\rho\mathbf{A}$ plus the divergence of a vector field, and that $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ is equal to $\mathbf{r} \times \rho\mathbf{A}$ plus the divergence of another vector field. Then, we transform the volume integrals of the auxiliary vector field into surface integral using Gauss' law, and if the auxiliary field fall off sufficiently quickly with distance the equivalence of the various forms of electromagnetic momenta is established.¹⁰

⁹The results of this section are also discussed for the case of axial symmetry in [27], without awareness of the prior work of [19, 25] or that axial symmetry is not required.

¹⁰The case of a charged particle together with an infinite magnetic solenoid is delicate in this regard, as discussed in [28, 29]. Here, $\mathbf{P}_{\text{EM}_P} = \mathbf{P}_{\text{EM}_M}$, but \mathbf{L}_{EM_M} rather than \mathbf{L}_{EM_P} yields reasonable physical results.

In addition to well-known vector-calculus relations, it is useful to define a combined operation

$$\nabla \cdot \mathbf{a}\mathbf{b} \equiv (\nabla \cdot \mathbf{a})\mathbf{b} + (\mathbf{a} \cdot \nabla)\mathbf{b} = (\nabla \cdot b_x \mathbf{a}) \hat{\mathbf{x}} + (\nabla \cdot b_y \mathbf{a}) \hat{\mathbf{y}} + (\nabla \cdot b_z \mathbf{a}) \hat{\mathbf{z}}. \quad (17)$$

Then,

$$\begin{aligned} \mathbf{E} \times \mathbf{B} &= \mathbf{E} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{A} \cdot \mathbf{E}) - (\mathbf{A} \cdot \nabla)\mathbf{E} - (\mathbf{E} \cdot \nabla)\mathbf{A} - \mathbf{A} \times (\nabla \times \mathbf{E}) \\ &= (\nabla \cdot \mathbf{E})\mathbf{A} + \nabla(\mathbf{A} \cdot \mathbf{E}) - [(\nabla \cdot \mathbf{E})\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{E}] - [(\nabla \cdot \mathbf{A})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{A}] \\ &= 4\pi\rho\mathbf{A} + \nabla(\mathbf{A} \cdot \mathbf{E}) - \nabla \cdot \mathbf{E}\mathbf{A} - \nabla \cdot \mathbf{A}\mathbf{E}, \end{aligned} \quad (18)$$

so that

$$\begin{aligned} \mathbf{P}_{\text{EMP}} &= \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} \\ &= \int \frac{\rho\mathbf{A}}{c} d\text{Vol} + \oint (\mathbf{A} \cdot \mathbf{E}) d\text{Area} - \oint \mathbf{E}(\mathbf{A} \cdot d\text{Area}) - \oint \mathbf{A}(\mathbf{E} \cdot d\text{Area}) \\ &= \int \frac{\rho\mathbf{A}}{c} d\text{Vol} = \mathbf{P}_{\text{EMM}}. \end{aligned} \quad (19)$$

The surface integrals in eq. (19) are negligible when the charges and currents that create the electric field \mathbf{E} and the vector potential \mathbf{A} lie within a finite volume that is small compared to the volume of integration, and when radiation can be neglected.

We now evaluate eq. (16) by taking the cross product of eq. (18) with \mathbf{r} and further transforming the various terms. Thus,

$$\mathbf{r} \times \nabla(\mathbf{A} \cdot \mathbf{E}) = -\nabla \times (\mathbf{A} \cdot \mathbf{E})\mathbf{r} + \nabla(\mathbf{A} \cdot \mathbf{E})\nabla \times \mathbf{r} = -\nabla \times (\mathbf{A} \cdot \mathbf{E})\mathbf{r}. \quad (20)$$

Now,

$$\begin{aligned} [\mathbf{r} \times \nabla \cdot \mathbf{E}\mathbf{A}]_x &= y\nabla \cdot (A_z\mathbf{E}) - z\nabla \cdot (A_y\mathbf{E}) \\ &= \nabla \cdot (yA_z\mathbf{E}) - A_z\mathbf{E} \cdot \nabla y - \nabla \cdot (zA_y\mathbf{E}) + A_y\mathbf{E} \cdot \nabla z \\ &= A_yE_z - A_zE_y + \nabla \cdot (yA_z\mathbf{E}) - \nabla \cdot (zA_y\mathbf{E}) \\ &= [\mathbf{A} \times \mathbf{E}]_x + \nabla \cdot ([\mathbf{r} \times \mathbf{A}]_x\mathbf{E}), \end{aligned} \quad (21)$$

so

$$[\mathbf{r} \times \nabla \cdot \mathbf{A}\mathbf{E}]_x = [\mathbf{E} \times \mathbf{A}]_x + \nabla \cdot ([\mathbf{r} \times \mathbf{E}]_x\mathbf{A}), \quad (22)$$

and

$$[\mathbf{r} \times \nabla \cdot \mathbf{E}\mathbf{A}]_x + [\mathbf{r} \times \nabla \cdot \mathbf{A}\mathbf{E}]_x = \nabla \cdot ([\mathbf{r} \times \mathbf{A}]_x\mathbf{E}) + \nabla \cdot ([\mathbf{r} \times \mathbf{E}]_x\mathbf{A}). \quad (23)$$

Hence,

$$\begin{aligned} \mathbf{L}_{\text{EMP}} &= \int \frac{\mathbf{r} \times (\mathbf{E} \times \mathbf{B})}{4\pi c} d\text{Vol} \\ &= \int \mathbf{r} \times \frac{\rho\mathbf{A}}{c} d\text{Vol} - \oint (\mathbf{A} \cdot \mathbf{E}) d\text{Area} \times \mathbf{r} - \oint \mathbf{r} \times \mathbf{E}(\mathbf{A} \cdot d\text{Area}) - \oint \mathbf{r} \times \mathbf{A}(\mathbf{E} \cdot d\text{Area}) \\ &= \int \mathbf{r} \times \frac{\rho\mathbf{A}}{c} d\text{Vol} = \mathbf{L}_{\text{EMM}}. \end{aligned} \quad (24)$$

The arguments of the surface integrals in eq. (24) vanish less quickly with distance than those in eq. (19), so in some cases¹¹ with sources of infinite extent, we may find that $\mathbf{P}_{\text{EMP}} = \mathbf{P}_{\text{EMM}}$,

¹¹See, for example, [28, 29].

but $\mathbf{L}_{\text{EM}_P} \neq \mathbf{L}_{\text{EM}_M}$.

The equivalence of \mathbf{P}_{EM_P} and \mathbf{P}_{EM_M} extends to nonstatic systems in which the currents do not satisfy $\nabla \cdot \mathbf{J} = 0$, so long as the velocities of all charges are low and radiation can be neglected [30, 31].

2.3 Equivalence of the Faraday-Maxwell and Furry Forms

In (quasi)static situations (where $\nabla \cdot \mathbf{A} = 0$) the potentials are related to the charge and current densities by

$$\nabla^2 V = 4\pi\rho, \quad \nabla^2 \mathbf{A} = \frac{4\pi\mathbf{J}}{c}. \quad (25)$$

Hence, we can start from the Faraday-Maxwell form of the electromagnetic momentum and write

$$\begin{aligned} \mathbf{P}_{\text{EM}_M} &= \int \frac{\rho\mathbf{A}}{c} d\text{Vol} = \int \frac{\mathbf{A}\nabla^2 V}{4\pi c} d\text{Vol} \\ &= \int \frac{V\nabla^2 \mathbf{A}}{4\pi c} d\text{Vol} + \oint [\mathbf{A}(d\mathbf{Area} \cdot \nabla)V - V(d\mathbf{Area} \cdot \nabla)\mathbf{A}] \\ &= \int \frac{V\mathbf{J}}{c^2} d\text{Vol} = \mathbf{P}_{\text{EM}_F}, \end{aligned} \quad (26)$$

using Green's identity that for any two well-behaved scalar fields ϕ and ψ ,

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi) d\text{Vol} = \oint_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\mathbf{Area}, \quad (27)$$

where the surface element $d\mathbf{Area}$ is directly away from the closed surface S that bounds volume V . The integral in eq. (26) over the surface at infinity vanishes for bounded charge and current densities, whose corresponding potentials fall off as $1/r$, and whose gradients fall off as $1/r^2$, at large r .

Another derivation of \mathbf{P}_{EM_F} notes that since the magnetic field \mathbf{B} is always of order $1/c$ (or higher), we can calculate the electromagnetic momentum \mathbf{P}_{EM_M} to order $1/c^2$ using approximations to the electric field at zeroth order, *i.e.*, $\mathbf{E} \approx -\nabla V^{(C)}$ (the Coulomb electric field), and to the magnetic field at order $1/c$, *i.e.*, $\nabla \times \mathbf{B} \approx 4\pi\mathbf{J}/c$ with the neglect of the displacement current. Then,

$$\begin{aligned} \mathbf{P}_{\text{EM}_P} &= \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} \approx - \int \frac{\nabla V^{(C)} \times \mathbf{B}}{4\pi c} d\text{Vol} \\ &= \int \frac{V^{(C)}\nabla \times \mathbf{B}}{4\pi c} d\text{Vol} - \int \frac{\nabla \times V^{(C)}\mathbf{B}}{4\pi c} d\text{Vol} \\ &= \int \frac{V^{(C)}\mathbf{J}}{c^2} d\text{Vol} - \oint \frac{d\mathbf{Area} \times V^{(C)}\mathbf{B}}{4\pi c} = \int \frac{V^{(C)}\mathbf{J}}{c^2} d\text{Vol} = \mathbf{P}_{\text{EM}_F}, \end{aligned} \quad (28)$$

whenever the charges and currents are contained within a finite volume.

However, the Furry density $V\mathbf{J}/c^2$ does not lead to a satisfactory computation of field angular momentum,

$$\mathbf{L}_{\text{EM}_F} = \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2} d\text{Vol} \neq \mathbf{L}_{\text{EM}_P}. \quad (29)$$

To see this, we note that for stationary systems where $\mathbf{E} = -\nabla V$, *i.e.*, $E_i = -\partial_i V$, and $\nabla \times \mathbf{B} = 4\pi\mathbf{J}/c$,

$$\begin{aligned} [\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_i &= \epsilon_{ijk} r_j \epsilon_{klm} (-\partial_l V) B_m = -\epsilon_{ijk} \epsilon_{klm} r_j [\partial_l (V B_m) - V \partial_l B_m] \\ &= -\epsilon_{ijk} \epsilon_{klm} [\partial_l (r_j V B_m) - V B_m \partial_l r_j] + \epsilon_{ijk} V r_j (\nabla \times \mathbf{B})_k \\ &\rightarrow \epsilon_{ijk} \epsilon_{klm} V B_m \delta_{jl} + \frac{4\pi}{c} \epsilon_{ijk} V r_j J_k = -\epsilon_{ijk} \epsilon_{mjk} V B_m + \frac{4\pi V}{c} (\mathbf{r} \times \mathbf{J})_i \\ &= -\delta_{ij}^{mj} V B_m + \frac{4\pi}{c} (\mathbf{r} \times V\mathbf{J})_i = -2\delta_{im} V B_m + \frac{4\pi}{c} (\mathbf{r} \times V\mathbf{J})_i \\ &= -2V B_i + \frac{4\pi}{c} (\mathbf{r} \times V\mathbf{J})_i, \end{aligned} \quad (30)$$

where in the third line we drop the full-differential term $\partial_l (r_j V B_m)$, anticipating that $\int \partial_l (r_j V B_m) d\text{Vol} \rightarrow \oint r_j V B_m d\text{Area} \rightarrow 0$ for fields that fall off sufficiently quickly at large distances. Then, for bounded, stationary systems, where $\mathbf{L}_P = \mathbf{L}_M$,

$$\mathbf{L}_{\text{EM}_P} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = - \int \frac{V\mathbf{B}}{2\pi c} d\text{Vol} + \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2} d\text{Vol} = \mathbf{L}_{\text{EM}_F} - \int \frac{V\mathbf{B}}{2\pi c} d\text{Vol}. \quad (31)$$

That is, the Poynting and Furry forms of field angular momentum are not equal, in general.¹²

2.4 Equivalence of the Aharonov and Furry Forms

For a stationary system, $\mathbf{E} = -\nabla V$ and $\nabla \cdot \mathbf{J} = \partial^j J_j = 0$, so

$$\mathbf{J} \cdot \mathbf{E} = -J_j \partial^j V = -\partial^j (J_j V) + V \partial^j J_j = -\partial^j (J_j V), \quad (32)$$

$$\begin{aligned} \mathbf{J} \cdot \mathbf{E} r_i &= -r_i J_j \partial^j V = -\partial^j (r_i J_j V) + V \partial^j (r_i J_j) = -\partial^j (r_i J_j V) + V J_j \partial^j r_i + V r_i \partial^j J_j \\ &= -\partial^j (r_i J_j V) + V J_i, \end{aligned} \quad (33)$$

and

$$P_{\text{EM}_A, i} = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} r_i d\text{Vol} = \int \frac{V J_i}{c^2} d\text{Vol} - \frac{1}{c^2} \int \partial^j (r_i J_j V) d\text{Vol}. \quad (34)$$

The latter volume integral becomes a surface integral at infinity, so is zero for any bounded current density. Then,

$$\mathbf{P}_{\text{EM}_A} = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} d\text{Vol} = \int \frac{V\mathbf{J}}{c^2} d\text{Vol} = \mathbf{P}_{\text{EM}_F}. \quad (35)$$

¹²For stationary fields that fall off sufficiently quickly at large distance, $\int V\mathbf{B} d\text{Vol} = \int \mathbf{E} \times \mathbf{A} d\text{Vol}$.

We could also note that for stationary systems, $\nabla \times \mathbf{E} = 0$, $\mathbf{J} = (c/4\pi)\nabla \times \mathbf{B}$, $J_j = (c/4\pi)\epsilon_{jkl}\partial_k B_l$, so

$$\begin{aligned}
\mathbf{J} \cdot \mathbf{E} r_i &= \frac{c}{4\pi} r_i E_j \epsilon_{jkl} \partial_k B_l = \frac{c}{4\pi} \partial_k (r_i \epsilon_{jkl} E_j B_l) - \frac{c}{4\pi} \epsilon_{jkl} E_j B_l \partial_k r_i - \frac{c}{4\pi} \epsilon_{jkl} r_i B_l \partial_k E_j \\
&= -\frac{c}{4\pi} \partial_k (r_i \epsilon_{kjl} E_j B_l) - \frac{c}{4\pi} \epsilon_{jkl} E_j B_l \delta_{ik} + \frac{c}{4\pi} r_i B_l \epsilon_{lkj} \partial_k E_j \\
&= -\frac{c}{4\pi} \partial_k [r_i (\mathbf{E} \times \mathbf{B})_k] + \frac{c}{4\pi} \epsilon_{ijl} E_j B_l + \frac{c}{4\pi} r_i B_l (\nabla \times \mathbf{E})_l \\
&= -\frac{c}{4\pi} \partial_k [r_i (\mathbf{E} \times \mathbf{B})_k] + \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B})_i,
\end{aligned} \tag{36}$$

and

$$P_{\text{EM}_A, i} = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} r_i d\text{Vol} = -\frac{1}{4\pi c} \int \partial_k [r_i (\mathbf{E} \times \mathbf{B})_k] d\text{Vol} + \int \frac{(\mathbf{E} \times \mathbf{B})_i}{4\pi c} d\text{Vol}. \tag{37}$$

The volume integral involving ∂_k transforms to a surface integral at infinity, which vanishes for fields that fall off suitably quickly, and we have that

$$\mathbf{P}_{\text{EM}_A} = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \mathbf{P}_{\text{EM}_P}. \tag{38}$$

However, the moment $\mathbf{r} \times \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r}$ of the Aharonov density $\frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r}$ is identically zero, so the field angular momentum cannot be computed via this form,

$$\mathbf{L}_{\text{EM}_P} \neq \mathbf{L}_{\text{EM}_A} = \int \mathbf{r} \times \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} d\text{Vol} = 0. \tag{39}$$

A Appendix: Rotating Charge Distributions

In quasistatic examples in which all electric charges lie on uniformly charged cylindrical shells that rotate at constant angular velocity $\boldsymbol{\omega}$ about a common axis, the electric and magnetic fields have the forms $\mathbf{E} = E(r_\perp) \hat{\mathbf{r}}_\perp$ and $\mathbf{B} = B(r_\perp) \hat{\boldsymbol{\omega}}$, where \mathbf{r}_\perp is perpendicular to $\boldsymbol{\omega}$. It is tempting to suppose that the lines of the electric and magnetic fields also rotate with angular velocity $\boldsymbol{\omega}$.

Following Einstein [32], we can identify an effective mass density

$$\rho_{\text{eff}} = \frac{E^2 + B^2}{8\pi c^2}, \tag{40}$$

with the density $(E^2 + B^2)/8\pi$ of energy in the electromagnetic fields. Choosing the origin to be on the axis of rotation, the velocity of rotation of a point at location \mathbf{r} is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. This suggests that we identify densities of momentum and angular momentum in the supposedly rotating electromagnetic fields as

$$\mathbf{p}_{\text{EM,eff}} \stackrel{?}{=} \rho_{\text{eff}} \mathbf{v} = \frac{E^2 + B^2}{8\pi c^2} \boldsymbol{\omega} \times \mathbf{r}, \quad \text{and} \quad \mathbf{l}_{\text{EM,eff}} \stackrel{?}{=} \mathbf{r} \times \rho_{\text{eff}} \mathbf{v} = \mathbf{r} \times \frac{E^2 + B^2}{8\pi c^2} (\boldsymbol{\omega} \times \mathbf{r}). \tag{41}$$

However, in examples where the field lines extend to large distances, the velocity of the rotating field lines can exceed the speed of light, such that it is implausible to associate them with physical momenta and angular momenta.

Indeed, in such examples, the field angular momentum is infinite when computed via the above hypothesis. And in examples where the field lines lie only within bounded volumes, the field angular momentum so computed does not agree with that calculated via the standard form (16) [33].

Of course, Faraday long ago concluded that the magnetic field lines do not rotate in such examples.¹³

The argument of this Appendix is that it's also best to consider that the electric field lines associated with rotating cylinders of charge do **not** rotate along with the charge.

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