Bunch-Timing Measurement
in the Muon Cooling Experiment
Via a Rectangular TM$_{2,1,0}$
RF Deflection Cavity

1 Introduction

In the previous note, Princeton/$\mu\mu$/97-5, we studied the transverse displacement of a muon on passing through a rectangular TE$_{0,1,1}$ RF cavity as a function of longitudinal position within the bunch. However, the effect was rather small. Here we consider a rectangular TM$_{2,1,0}$ cavity in which a muon is deflected by the magnetic field, which is large at the center of the cavity. The angular deflection is about three times that of the effective angular deflection in a TE$_{0,1,1}$ cavity, which permits the tracking system and cavity wall to be about ten times thicker for the same time resolution.

2 Rectangular TM$_{2,1,0}$ Cavity Fields

The rf cavity is centered on $(x, y, z) = (0, 0, 0)$, and is a rectangular box of length $a$ in $x$ (the direction of transverse deflection), and length $a/\alpha$ in $y$ and length $b$ along the beam direction $z$.

The trajectory of a typical beam particle for the cavity field OFF is parametrized as

\begin{align}
  x &= x_0 + \beta_x ct, \\
  y &= y_0 + \beta_y ct, \\
  z &= z_0 + \beta_z ct,
\end{align}

where $c$ is the speed of light. The beam axis is the $z$-axis:

\begin{align}
  \beta_x, \beta_y \ll \beta_z, \quad \text{and} \quad \beta_z \approx \beta.
\end{align}

We will make the impulse approximation that the cavity fields do not affect the muon trajectories in $y$ or $z$, but only in $x$. Thus we assume the $y$ and $z$ parametrizations in (1) also hold when the field is ON.
The particle is within the cavity during the interval

$$[t_{\text{min}}, t_{\text{max}}] = \left[ -\frac{b}{2\beta_z c} - \frac{z_0}{\beta_z c}, \frac{b}{2\beta_z c} - \frac{z_0}{\beta_z c} \right].$$

(3)

The wave equation tells us that for a TM$_{2,1,0}$ cavity

$$\frac{\omega}{c} = \sqrt{4 + \alpha^2 \frac{\pi}{a}},$$

so

$$a = \frac{\sqrt{4 + \alpha^2}}{2} \lambda,$$

(4)

where $\omega$ is the angular frequency and $\lambda = c/2\pi \omega$ is the wavelength. For frequency $\nu = 800$ MHz, $\lambda = 37.5$ cm.

The cavity is phased so that the magnetic field is minimum at $t = 0$. In Gaussian units,

$$E_x = E_y = 0,$$
$$E_z = E_0 \sin \frac{2\pi x}{a} \cos \frac{\pi y}{a} \cos \omega t,$$
$$B_x = \frac{\alpha E_0}{\sqrt{4 + \alpha^2}} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{a} \sin \omega t,$$
$$B_y = \frac{2E_0}{\sqrt{4 + \alpha^2}} \cos \frac{2\pi x}{a} \cos \frac{\pi y}{a} \sin \omega t,$$
$$B_z = 0.$$  

(5)

### 3 Transverse Deflection: Leading Approximation

To a good approximation the energy of the muon does not change in the RF cavity; $\gamma = 1/\sqrt{1 - \beta^2}$ remains constant. The $x$-component of the Lorentz-force law can then be written

$$\frac{d\beta_x}{dt} = \frac{e}{\gamma mc} (E_x + \beta_y B_z - \beta_z B_y) = -\frac{\beta_x eB_y}{\gamma mc},$$

$$= -\frac{2\beta_z eE_0}{\gamma \sqrt{4 + \alpha^2 mc}} \cos \frac{2\pi x}{a} \cos \frac{\pi y}{a} \sin \omega t \approx -\frac{2\beta_z eE_0}{\gamma \sqrt{4 + \alpha^2 mc}} \sin \omega t,$$

(6)

using eq. (5) and supposing that the transverse size of the beam is small compared to $a$.

From eq. (6) we see that the angular kick is $x$ depends on the $x$ and $y$ positions and slopes only in second order.

The change in the $x$-velocity due to the RF cavity is

$$\Delta \beta_x = \int_{t_{\text{min}}}^{t_{\text{max}}} \frac{d\beta_x}{dt} dt \approx -\frac{2\beta_z eE_0}{\gamma \sqrt{4 + \alpha^2 mc}} \int_{t_{\text{min}}}^{t_{\text{max}}} \sin \omega t dt$$

$$= \frac{2\beta_z eE_0}{\gamma \sqrt{4 + \alpha^2 mc}} \frac{\cos \omega t_{\text{max}} - \cos \omega t_{\text{min}}}{\omega} \approx \frac{8\pi}{\gamma \sqrt{4 + \alpha^2}} \frac{\eta z_0}{\lambda} \sin \frac{\omega b}{2\beta_z c},$$

(7)

using eq. (3) and introducing the dimensionless measure of field strength

$$\eta = \frac{eE_0}{m\omega c} = \frac{eE_0}{mc^2 2\pi \nu}.$$  

(8)
We choose length $b$ so that the argument of the sine is $\pi/2$:

$$b = \frac{\pi \beta c}{\omega} = \frac{\beta \lambda}{2} = 15.75 \text{ cm}$$  \hspace{1cm} (9)

for $\nu = 800 \text{ MHz}$ and $\beta_z = 0.84$, corresponding to muon momentum of 165 MeV/c. Thus a muon is in the cavity for half a cycle. Then,

$$\Delta \beta_z \approx \frac{8\pi}{\gamma \sqrt{4 + \alpha^2}} \eta z_0.$$  \hspace{1cm} (10)

The corresponding angular deflection is

$$\Delta \theta_x = \frac{\Delta \beta_z}{\beta_z} \approx \frac{8\pi}{\gamma \beta_z \sqrt{4 + \alpha^2}} \eta z_0 = \frac{8\pi}{\gamma \sqrt{4 + \alpha^2}} \eta c \Delta t,$$  \hspace{1cm} (11)

introducing the time offset $\Delta t = z_0/\beta_z c$ of the muon from the center of the bunch.

4 Discussion

To maintain $x$-$y$ symmetry we consider aspect ratio $\alpha = 2$, for which $a = 53 \text{ cm}$ at 800 MHz, and

$$\Delta \theta_x \approx \frac{2\sqrt{2}\pi \eta c \Delta t}{\lambda}.$$  \hspace{1cm} (12)

As an example we consider a peak field $E_0 = 40 \text{ MV/m}$, corresponding to $\eta = 0.0223$, and 165-MeV/c muons for which $\gamma = 1.85$. Then

$$\Delta \theta_x \approx \frac{2\sqrt{2}\pi \cdot 0.0223 \cdot 3 \times 10^{-4} \text{ m/ps}}{1.85 \cdot 0.375 \text{ m}} \Delta t[\text{ps}] = 86 \mu \text{rad}.$$  \hspace{1cm} (13)

This is nearly three times the angular deflection found for a TE$_{0,1,1}$ cavity (eq. (63) in our note Princeton/$\mu\mu$-97-5). If we used $\alpha = 1$ the angular deflection would be 26% larger.

It will be difficult the measure the small angular deflections the arise from a single cavity. However, if we make a multicell RF deflection structure (with irises to let the beam pass) the angular deflection would be cumulative, to our advantage.

Continuing the analysis of a single cavity, we suppose the timing resolution function is well understood and so we relax the requirement on timing resolution to, say $\sigma_{t,D} = 0.5 \sigma_t = 20 \text{ ps}$, following Table 1 of our note Princeton/$\mu\mu$/97-4. This requires knowing the timing resolution function to 4%. Then the corresponding angular resolution is $\sigma_{\theta_x,D} = 1.7 \text{ mrad}$. Then the material in the cavity wall and tracking system just upstream must satisfy

$$X_0 < \left( \frac{\sigma_{\theta_x,D} P \beta_z}{15 \text{ MeV/c}} \right)^2 = \left( \frac{0.0017 \cdot 165 \cdot 0.84}{15} \right)^2 = 0.00025 \text{ radiation lengths}.$$  \hspace{1cm} (14)

The cavity wall could then be about 75 $\mu\text{m}$ thick if made of beryllium.