Learning Guide No. 1: One-Dimensional Motion, Vectors, Motion in Two and Three Dimensions

Suggested Reading: Tipler, Chapters 1, 2, and 3

**Problem I**

A student is rowing a rowboat on a river whose current flows with speed \( v \). She can row at a speed \( V \) with respect to the water. Since \( V > v \) she can row upstream as well as downstream. She decides to row a distance \( d \) (relative to the shoreline) upstream and then turn around and row downstream the same distance \( d \) to her starting point.

1. How long does it take the student to complete the round trip? If your answer doesn’t check even after a good effort, look at Helping Questions 1 and 2.  
   Key 10

2. Does the trip take longer when \( v = 0 \) or when \( v > 0 \)? Why?  
   Key 12

**Problem II**

A stone is dropped from the roof of an 80-m-high building. Its displacement, \( y \), measured from the point of release is given by \( y(t) = 5t^2 \), where \( y \) is in meters and the time \( t \) is in seconds. In this problem you will calculate the stone’s acceleration in two ways: by averaging over short time intervals, and by using calculus.

1. What is the displacement of the stone at \( t = 0, 1, 2, 3, \) and 4 s? What are the average velocities in the intervals 0–1 s, 1–2 s, 2–3 s, and 3–4 s? For small intervals, the instantaneous velocity at the center of an interval is approximately equal to the average velocity over the interval. Assuming for sake of argument that these two quantities are equal, what is the average acceleration in the time intervals 0.5–1.5 s, 1.5–2.5 s, and 2.5–3.5 s? If you’re having trouble, use Helping Question 3.  
   Key 25

2. Use calculus (i.e., take a derivative) to find the instantaneous velocity as a function of time. Find the instantaneous acceleration as a function of time. If you are stuck, see Helping Question 4.  
   Key 21

3. If the physical situation had been different so that the displacement function was \( y(t) = 5t^4 \) instead of \( y(t) = 5t^2 \), which method would have been more accurate for calculating the instantaneous acceleration?  
   Key 6
Problem III

Vector Warm-Ups

1. Let \( \mathbf{A} = 3\mathbf{i} + 4\mathbf{j} \) and let \( \mathbf{B} = 5\mathbf{i} + 12\mathbf{j} \). What is the magnitude of \( \mathbf{A} \)? What is the magnitude of \( \mathbf{B} \)? What is \( \mathbf{A} + \mathbf{B} \)? What is the angle between \( \mathbf{A} \) and \( \mathbf{B} \)? If you’re having trouble, look at Tipler Sections 3-1 and 3-2.  

   Key 19

2. Now let \( \mathbf{A} = 3\mathbf{i} - 4\mathbf{j} \) and let \( \mathbf{B} = -5\mathbf{i} - 12\mathbf{j} \). What is the magnitude of \( \mathbf{A} \)? What is the magnitude of \( \mathbf{B} \)? What is \( \mathbf{A} + \mathbf{B} \)? What is the angle between \( \mathbf{A} \) and \( \mathbf{B} \)?  

   Key 9

3. Show the region in the \( xy \)-plane that contains the end points of all possible vectors \( \mathbf{A} + \mathbf{B} \) where \( \mathbf{A} \) is a vector of magnitude 5 and \( \mathbf{B} \) is a vector of magnitude 13.  

   Key 1

Problem IV

An amateur football player punts a football straight up at \( v_0 = 15 \text{ m/s} \). Assume that the acceleration due to gravity is \( g = 10 \text{ m/s}^2 \).

1. How long is the football in the air? If you need help getting started, use Helping Questions 5 and 6.  

   Key 26

2. How high does the football go? Helping Question 7 will put you on the right track.  

   Key 7

Next, a professional football player punts a football straight up. The football leaves at twice the speed of the amateur player’s football; i.e., with \( v_0 = 30 \text{ m/s} \).

3. Does the ball stay in the air twice as long?  

   Key 11

4. Does it go twice as high?  

   Key 3

Problem V

A grazing antelope first notices a lion attacking when the lion is 12.5 m away and moving toward the antelope at a speed of 5 m/s. The antelope begins to accelerate away from the lion at 3 m/s\(^2\) and the lion simultaneously begins to accelerate at 2 m/s\(^2\).

1. How long does the antelope’s flight last?  

   Key 15
2. How far has the antelope traveled when the lion catches up with it?  

If you need help see Helping Question 8. 

**Problem VI**

An object dropped from a stationary helicopter falls straight down toward the earth. At the beginning of its flight, its acceleration is $g$, the acceleration due to gravity. As its speed increases, it meets with increasing amounts of air resistance. To a reasonable approximation, its motion satisfies $a(t) = g - kv(t)$ where $a(t)$ and $v(t)$ are the instantaneous acceleration and velocity and $k$ is a positive constant that depends on the shape and surface roughness of the object. Without solving an equation for $v(t)$ use physical reasoning to sketch a graph of speed versus time. You should discover the terminal velocity phenomenon: the object never accelerates past a particular speed, called the terminal velocity. What is the terminal velocity for the object in terms of $g$ and $k$? Turn to Helping Questions 9 and 10 if you need help. 

**Problem VII**

In Problem II, you were given $x(t)$ and asked to find $v(t)$ and $a(t)$. Suppose you were given $a(t)$ and asked to find $v(t)$ and then $x(t)$. In this problem, you will learn a geometrical solution to this question. You know that:

$$v(t) = v_0 + at$$

for motion with constant acceleration. Referring to the graph of $a(t)$ in the figure, note that:

$$v(t) = v_0 + \text{area under the curve between 0 and } t,$$

since the shaded area is precisely $at$. In fact, it turns out that equation (1) is generally true, even if the acceleration is not constant. In such cases, the equation $v = v_0 + at$ no longer makes sense (since $a$ is no longer a single number).
As an example, consider the next sketch, where $a(t)$ varies with time. The area of the shaded region is about 6 m/s, and so the velocity at $t = 3$ s is $v_0 + 6$ m/s.

In the same way, the equation of motion $x(t) = x_0 + vt$ not only applies for motion at constant $v$ but generalizes to:

$$x(t) = x_0 + \text{area under the } v(t) \text{ curve between 0 and } t$$

no matter how the velocity changes with time.

For both the acceleration and the velocity curves, a slight complication enters if the curves go beneath the horizontal axis — i.e., if the acceleration or velocity goes negative. Then “area under the curve” must be replaced by “area above the horizontal axis minus the area below the horizontal axis,” as shown in the sketch. You should convince yourself that this extension is physically reasonable. In the example shown, the position at $t = 3$ s is about $x(3) = x_0 + 1$ m. (Think about where the particle is at the top of the first “+” bump.)

For practice, consider a particle moving with acceleration given by the adjacent graph. Take $v_0 = 0$ and $x_0 = 0$. Graph as accurately as you can, putting numbers on the axes:

1. $v(t)$
2. $x(t)$
Problem VIII

A student can row a boat 8 km/h in still water. He is on one bank of an 8-km-wide river that has a current of 4 km/h.

1. What is the smallest amount of time he needs to get to the other bank, assuming that it’s acceptable to land anywhere on the other shore? If you’re confused, see Helping Question 11. \( \text{Key 41} \)

2. If the student wants to get to the point exactly across from where he starts, what angle should his boat make with respect to a line running straight across the river? Helping Question 12 will help you with this part. \( \text{Key 31} \)

3. Take the point of view of an observer on the shore. In your reference frame, the student’s velocity vector is the sum of a vector of length 8 km/h (his rowing speed in still water) and a vector of length 4 km/h (the speed of the river’s current). In your reference frame, what is the student’s speed in part (1)? In part (2)? \( \text{Key 34} \)

If you need some hints, turn to Helping Questions 13 and 14.

Problem IX

A cannon fires a projectile at an angle \( \theta \) with respect to the horizontal. The speed of the projectile as it leaves the cannon’s barrel is \( v_0 \). Find an expression that gives the horizontal range of the shell, \( R \), in terms of \( \theta \), \( v_0 \), and \( g \). See Helping Questions 15 and 16. (This is the range formula, which is also derived in Tipler, but don’t look there unless you’re really stuck.) \( \text{Key 40} \)
**Problem X**

A woman standing on a cliff of height $h$ has a baseball that she can throw with speed $v$. She wants to throw the baseball as far away from the cliff as she can. In terms of the variables indicated on the diagram, she wants to choose $\theta$ to maximize $d$. For this problem you may neglect the height of the woman, air resistance and the bouncing or rolling of the ball. Before you start, you might want to guess at $\theta_{\text{max}}$ for $h = 0$ and $h$ very high. **Note:** The range formula from Problem II does *not* apply. It is valid only in cases where the initial and final heights are the same.

1. Express $d$ in terms of $v$, $\theta$, $h$, and $g$, the acceleration due to gravity. Use Helping Questions 17 and 18. **Key 46**

2. Set $h = 0$ in your answer for part (1). What is $\theta_{\text{max}}$? Does $\theta_{\text{max}}$ depend on $v$? For a hint, see Helping Question 19. **Key 30**

3. Now suppose that $h$ is “infinitely large” so that terms not containing $h$ can be ignored. What is $\theta_{\text{max}}$? Does $\theta_{\text{max}}$ depend on $v$? Stumped? See Helping Question 20. **Key 29**

4. Finally, consider the case where $h$ is finite and positive. Between what two values of $\theta$ is $\theta_{\text{max}}$? Use your physical intuition to decide whether $\theta_{\text{max}}$ depends on $v$. Use Helping Question 21 for the last question. **Key 43**
HELPING QUESTIONS

1. What is the student’s speed relative to the shoreline when she rows upstream? When she rows downstream?  
   Key 24

2. How long does it take her to row upstream? Downstream?  
   Key 8

3. What is the definition of average velocity? Average acceleration?  
   Key 13

4. Define instantaneous velocity and acceleration mathematically.  
   Key 20

5. Can you think of a kinematic formula that relates what you know: the starting point \( y_0 = 0 \), the end point \( y = 0 \), \( v_0 \), and \( g \); to what you want to find — the time of landing \( t \)?  
   Key 22

6. What are the two roots of the equation \( v_0 t - \frac{1}{2} g t^2 = 0 \)?  
   Key 14

7. You know the time at which the ball hits the ground. Can you say right away the time at which the ball is at the high point of the path? If you don’t see the intuitive answer, again try to find the right equation. The key step is to think of what quantity has a special behavior at the top.  
   Key 18

8. What is true about the two animals’ positions at the time of interest?  
   Key 27

9. Is the velocity of the object increasing or decreasing? Is the acceleration of the object increasing or decreasing? What do the answers to these two questions mean in terms of the plot of \( v(t) \)?  
   Key 16

10. Suppose the object was falling with terminal velocity. What would the acceleration be?  
    Key 23

11. Does the speed of the river’s current affect the crossing time?  
    Key 33

12. What direction must the velocity vector (rowing velocity plus current velocity) be pointing? Make a sketch, showing all three vectors and the angle you’re after.  
    Key 39

13. One way you can add vectors is to introduce a coordinate system and add the vectors by component. What would be a convenient coordinate system here?  
    Key 32

14. If a vector \( \mathbf{v} \) makes an angle \( \theta \) with the \( x \)-axis, what is \( v_x \) in terms of \( |\mathbf{v}| \) and \( \theta \)? What is \( v_y \)?  
    Key 37

15. What is \( x(t) \)? What is \( y(t) \)? If you need another hint, move on to Helping Question 6.  
    Key 44

16. \( y(t) \) has two zeros, what is their significance?  
    Key 28

17. If you knew the time \( t \) the ball spends in the air, could you get \( d \) in terms of \( \theta, v, \) and \( t \)?  
    Key 42

18. Can you get the time of flight \( t \) from the vertical component of the motion? You’ll have to use the quadratic formula.  
    Key 35

19. A trigonometric identity says that \( 2 \sin \theta \cos \theta = \sin(2\theta) \). Can you get the maximum by thinking about the graph of \( \sin(2\theta) \)? Alternatively, you can use calculus, setting the
derivative of \( d \) with respect to \( \theta \) equal to zero.

20. What is the expression for \( d \) when the terms not containing \( h \) are ignored?  

21. Think of a child and a professional baseball player on a 5-m cliff. The baseball player has a much higher \( v \) of course. To the child, is the cliff extremely high or negligible? What about to the baseball player?  

**Notes: Dimensions**

You should read Chapter 1 of Tipler about units and dimensions very carefully. As this course progresses, you will appreciate more and more their statement that **the dimensions on one side of an equation must be the same as those on the other side.** In fact, in any legitimate physical equation the dimensions of **all the terms** must be the same. Perhaps more to the point, a look at the dimensions of a solution to a problem often provides a quick “sanity check” on its veracity. If, for example, you are asked to calculate a velocity or a speed, your answer had better have units of meters per second. Any other dimensions indicate that something has gone awry. Old pros employ this trick to check their work as standard practice.

One can even check dimensions in equations that contain derivatives. The dimensions of a derivative are identical to the dimensions of the corresponding fractions. Thus, for example, \( \frac{dx}{dt} \) has the same dimensions as \( x/t \). Later on you will learn how to use integrals in physics, but from the point of view of dimensions an integral is just like a multiplication. You will also become familiar with algebraic operations between vectors — vector addition and two different types of vector multiplication. Dimensionally, these operations are the same as normal addition and multiplication.

One can even use dimensional analysis to make “ballpark” estimates. To illustrate this, let’s try to solve Problem IV about punted footballs using only dimensional arguments and intuition. Part (1) asks for the time \( t \) that the ball is in the air. Intuitively, the greater the initial speed \( v_0 \), the greater the time of flight \( t \). Intuition also says that the greater gravity \( g \) is, the smaller \( t \) is. So a possible solution is:

\[
    t = \frac{v_0}{g} \left( \frac{[L]/[T]}{[L]/[T]^2} = \frac{1}{1/[T]} = [T] \right)
\]

This equation checks dimensionally, as the computation in the parentheses shows (in that computation, \([L]\) and \([T]\) denote quantities having dimensions of length and time, respectively). In fact, the exact solution is \( t = \frac{2v_0}{g} \). This is about as close as dimensional analysis can get since the “2” is dimensionless. This sort of result is typical of the kind of accuracy one can expect using this approach.

Part (2) asks for the maximum height \( h \) of the ball. Again, intuition suggests a possible solution:

\[
    h = \frac{v_0}{g} \left( \frac{[L]/[T]}{[L]/[T]^2} = [T] \neq [L] \right).
\]
But this equation does not check dimensionally since \([L] \neq [T]\). If you think about it, the only simple way to fix it up is to write:

\[
\frac{v_0^2}{g} \left( \frac{[L]/[T]^2}{[L]/[T]^2} \right) = \frac{[L]^2/[T]^2}{[L]/[T]^2} = [L],
\]

which does check dimensionally. Again, it is off from the exact expression, \(h = \frac{v_0^2}{2g}\), by a dimensionless factor of 2.

The real triumph of dimensional analysis in this example is the answers it gives for parts (3) and (4). Even though it is off by dimensionless factors, it still predicts correctly that if the initial speed is doubled, the time of flight is also doubled but the maximum height is multiplied by 4.
ANSWER KEY

1. Inner circle has radius 8
   Outer circle has radius 18

2. \[ v(t) (\text{m/s}) \]

3. No, it goes 4 times higher.

4. 37.5 m

5. \( v_t = g/k \)

6. The calculus method of part (2) would have been better. No matter how small the time intervals were made in part (1), the answer would still be a little off.

7. 11.25 m

8. \( d/(V - v) \) upstream; \( d/(V + v) \) downstream

9. \(|\mathbf{A}| = 5, |\mathbf{B}| = 13, \mathbf{A} + \mathbf{B} = -2i - 16j, \theta_{AB} = 59.5^\circ|\)

10. \[ \frac{d}{V - v} + \frac{d}{V + v} = \frac{2dV}{V^2 - v^2} \]

11. Yes

12. The trip takes longer when \( v > 0 \) because when \( v > 0 \) the student’s average speed is less than \( V \); she spends more than half the time at speed \( V - v \) and less than half the time at speed \( V + v \).

13. \[ v_{av} = \frac{\Delta y}{\Delta t}; \quad a_{av} = \frac{\Delta v}{\Delta t} \]

14. The starting time \( t = 0 \) and the landing time \( t = \frac{2v_0}{g} \).

15. 5 s

16. The speed is increasing but the acceleration is decreasing, so the slope of the speed curve will always be positive but will become less steep at larger \( t \).

17.
18. Intuitive method: It’s at the top at the midpoint of the trip, or \( t = 1.5 \) s. Systematic method: The key point is that the ball stops instantaneously at the top, i.e., \( v = 0 \). Use \( 0 = v_0 - gt \) to solve for the time to get to the top.

19. \(|A| = 5, |B| = 13, A + B = 8i + 16j, \theta_{AB} = 14.3^\circ\)

20. \( v = \frac{dy}{dt}; \quad a = \frac{dv}{dt} \)

21. \( v = 10t, a = 10, \) so the acceleration is constant at 10 m/s\(^2\).

22. \( y = y_0 + v_0t + \frac{1}{2}at^2; \quad a = -g \)

23. 0

24. \( V - v \) upstream; \( V + v \) downstream.

25. For \( t = 0, 1, 2, 3, 4 \) s:
   \( y = 0, 5, 20, 45, 80 \) m
   \( v_{av} = 5, 15, 25, 35 \) m/s
   \( a_{av} = 10, 10, 10 \) m/s\(^2\)

26. 3 s

27. They are equal, i.e. at the time of interest \( x_{lion}(t) = x_{antelope}(t) \)

28. Launch time and landing time

29. \( \theta_{max} = 0^\circ, \) independent of \( v \).

30. \( \theta_{max} = 45^\circ, \) independent of \( v \).

31. \( 30^\circ \) upstream

32. Let the \( x \)-direction be downstream and the \( y \)-direction be across the river.

33. No

34. \( \sqrt{80} \simeq 8.94 \) km/h for part (1); \( \sqrt{48} \simeq 6.93 \) km/h for part (2).

35. \( t = \left( \frac{v \sin \theta}{g} + \frac{1}{g} \sqrt{v^2 \sin^2 \theta + 2gh} \right) \)

36. \( d = v \cos \theta \sqrt{\frac{2h}{g}} \)

37. \( v_x = |v| \cos \theta; \quad v_y = |v| \sin \theta \)

38. \( \sin(2\theta) \) has a maximum at \( \theta = 45^\circ \).

39. Straight across the river

40. \( R = \frac{2v_0^2}{g} \sin \theta \cos \theta = \frac{v_0^2}{g} \sin(2\theta) \)

41. 1 h

42. \( d = (v \cos \theta) t \)

43. \( \theta_{max} \) is between \( 0^\circ \) and \( 45^\circ \) and depends on \( v \).

44. \( x(t) = (v_0 \cos \theta) t \)
   \( y(t) = (v_0 \sin \theta) t - \frac{1}{2}gt^2 \)

45. The cliff is high to the child, and negligible to the player.

46. \( d = v \cos \theta \times \)
   \( \left( \frac{v \sin \theta}{g} + \sqrt{\frac{v^2 \sin^2 \theta}{g^2} + \frac{2h}{g}} \right) \)