PH 205 LECTURE 15

NON-LINEAR OSCILLATIONS

WE HAVE ESTABLISHED A METHOD FOR DESCRIBING SMALL OSCILLATIONS ABOUT EQUILIBRIUM OF ANY SYSTEM, BY EXPANDING THE POTENTIAL ENERGY AS

$$V(x) = V(0) + \frac{1}{2} V''(0)\, x^2 + \frac{1}{6} V'''(0)\, x^3 + \dots$$

AND KEEPING ONLY TERMS UP TO x^2: $V(x) \sim V(0) + \frac{1}{2} K x^2$

HENCE SIMPLE HARMONIC MOTION ALWAYS OCCURS IN THIS APPROXIMATION.

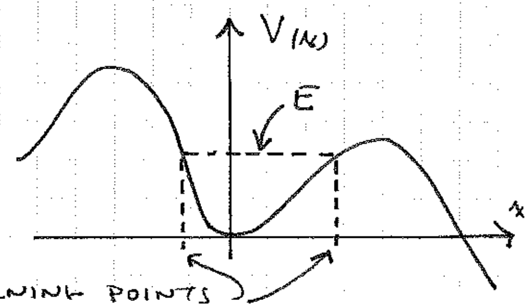
WHAT HAPPENS FOR LARGE DISPLACEMENTS FROM EQUILIBRIUM?

WE MUST KEEP MORE TERMS IN THE EXPANSION OF THE POTENTIAL.

THE FORCE IS NOW $F(x) = -\frac{dV}{dx} = -Kx - \frac{1}{2} V'''(0)\, x^2 - \frac{1}{6} V''''(0)\, x^3 - \dots$

WHICH IS NON-LINEAR IN x.

HOWEVER, SO LONG AS THE TOTAL ENERGY E IS NOT TOO HIGH, THE MOTION IS CONFINED BETWEEN THE TURNING POINTS, AND THE MOTION IS STILL OSCILLATORY.



HENCE THE PROBLEM OF NON-LINEAR OSCILLATIONS.

WE HAVE ALREADY GIVEN A FORMAL SOLUTION FOR THE ONE-DIMENSIONAL CASE:

$$\frac{1}{2} m \dot{x}^2 = E - V(x)$$

SO $\dot{x} = \sqrt{\frac{2}{m}(E - V(x))}$ AND $t = \int \frac{dx}{\sqrt{\frac{2}{m}(E - V)}}$

WE APPLIED THIS TO THE OSCILLATION OF A SIMPLE PENDULUM WITH LARGE DISPLACEMENT, AND FOUND THE PERIOD

$$T = 2\pi \sqrt{\frac{l}{g}} \left(1 + \frac{\Theta_{MAX}^2}{16} + \dots \right)$$

WE NOW WISH TO EXPLORE METHODS OF SOLUTION OF THIS NON-LINEAR OSCILLATION PROBLEM IN WHICH THE NON-LINEAR FORCE TERMS ARE REGARDED AS A CORRECTION. OUR SOLUTIONS WILL BE SIMPLE HARMONIC MOTION PLUS A CORRECTION.

THE METHOD OF SUCCESSIVE APPROXIMATIONS (L&L SEC 28)

WRITE $$V(x) = V(0) + \frac{1}{2} K x^2 + \frac{\alpha M}{3} x^3 + \frac{\beta M}{4} x^4 + \cdots$$

$$F(x) = -Kx - \alpha M x^2 - \beta M x^3 - \cdots$$

CASE I. IGNORE TERMS x^4 AND HIGHER IN THE POTENTIAL.

$$F(x) = -Kx - \alpha M x^2$$

OUR DIFFERENTIAL EQUATION IS (NO DAMPING)

$$\ddot{x} + \omega_0^2 x = -\alpha x^2 \qquad \omega_0^2 \equiv K/M$$

THE FIRST APPROXIMATION SOLUTION IS $(\alpha = 0)$

$$x = A \cos \omega_0 t$$

THE METHOD OF SUCCESSIVE APPROXIMATIONS IS TO SUPPOSE OUR COMPLETE SOLUTION IS OF THE FORM

$$x = A \cos \omega_0 t + \alpha\, x_1(t) + \alpha^2 x_2(t) + \cdots$$

WE SUBSTITUTE THIS TRIAL SOLUTION INTO THE DIFFERENTIAL EQUATION, AND EQUATE SEPARATELY THE COEFFICIENTS OF EACH POWER OF α. (REGARDING α AS AN ARBITRARY PARAMETER, SUCH A PROCEDURE MUST 'CLEARLY' HOLD.)

KEEPING ONLY TERMS UP TO α^2, WE GET

$$\alpha(\ddot{x}_1 + \omega_0^2 x_1) + \alpha^2(\ddot{x}_2 + \omega_0^2 x_2) = -\alpha\left(A^2 \cos^2 \omega_0 t + 2\alpha A x_1 \cos \omega_0 t\right)$$

HENCE
$$\ddot{x}_1 + \omega_0^2 x_1 = -A^2 \cos^2 \omega_0 t$$
$$\ddot{x}_2 + \omega_0^2 x_2 = -2A x_1 \cos \omega_0 t$$

THESE ARE JUST EQUATIONS OF FORCED HARMONIC MOTION

$$\ddot{x}_1 + \omega_0^2 x_1 = -\frac{A^2}{2}(1 + \cos 2\omega_0 t)$$

$$\Rightarrow \quad x_1 = \frac{-A^2}{2\omega_0^2} - \frac{A^2}{2} \frac{\cos 2\omega_0 t}{\omega_0^2 - 4\omega_0^2} = -\frac{A^2}{2\omega_0^2}\left(1 - \frac{\cos 2\omega_0 t}{3}\right)$$

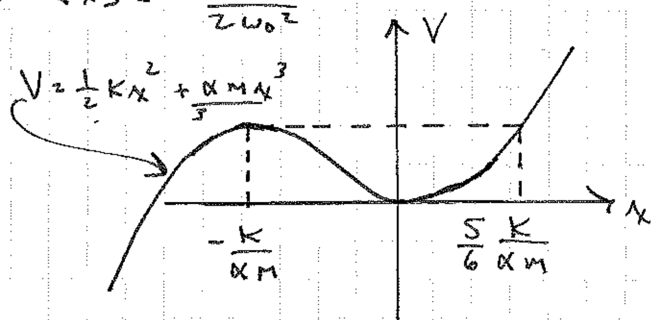
HENCE OUR SECOND APPROXIMATION TO THE MOTION IS

$$x = -\frac{\alpha A^2}{2\omega_0^2} + A \cos \omega_0 t + \frac{\alpha A^2}{6\omega_0^2} \cos 2\omega_0 t$$

WE NOTE 2 IMPORTANT FEATURES OF THE MOTION:

1) A SHIFT IN THE AVERAGE POSITION $\langle x \rangle = \frac{-\alpha A^2}{2\omega_0^2}$

THIS IS BECAUSE THE POTENTIAL ENERGY CURVE NOW OFFERS MORE ROOM AT $x<0$ THAN $x>0$. SO FOR A FIXED TOTAL $E>0$, THE PARTICLE SPENDS MORE TIME AT $x<0 \Rightarrow \langle x \rangle < 0$



2) THE OSCILLATORY MOTION CONTAINS A <u>HIGHER HARMONIC</u> $\cos 2\omega_0 t$.

THIS IS A MEMORABLE CHARACTERISTIC OF NON-LINEAR OSCILLATIONS. IF WE MAKE FURTHER APPROXIMATIONS: $x = A\cos\omega_0 t + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 \cdots$ WE WILL FIND MORE AND MORE HARMONICS: $\omega_0, 2\omega_0, 3\omega_0, 4\omega_0 \cdots$

HOWEVER IN THE PRESENT CASE WE ENCOUNTER A DIFFICULTY IN MAKING THE NEXT APPROXIMATION. FROM p 157, WE HAVE

$$\ddot{x}_2 + \omega_0^2 x_2 = -2Ax_1\cos\omega_0 t = \frac{A^3}{\omega_0^2}\cos\omega_0 t - \frac{A^3}{3\omega_0^2}\cos 2\omega_0 t \cos\omega_0 t$$

$$= \frac{5}{6}\frac{A^3}{\omega_0^2}\cos\omega_0 t - \frac{A^3}{6\omega_0^2}\cos 3\omega_0 t$$

THE $\cos 3\omega_0 t$ DRIVING TERM WILL LEAD TO 3RD HARMONIC MOTION AS ADVERTISED. BUT THE $\cos\omega_0 t$ TERM WILL LEAD TO RESONANCE! $\Rightarrow x_2 \to \infty$ (NO DAMPING) — NOT A 'SMALL' CORRECTION.

WE ILLUSTRATE THE WAY OUT OF THIS DIFFICULTY IN THE FOLLOWING EXAMPLE.

<u>CASE 2</u> $\quad V(x) = \frac{1}{2}kx^2 + \frac{\beta m}{4}x^4 \Longleftrightarrow F(x) = -kx - \beta m x^3$

THIS EXAMPLE CORRESPONDS TO THE SIMPLE PENDULUM, AS DISCUSSED FURTHER BELOW.

THE DIFFERENTIAL EQUATION IS $\ddot{x} + \omega_0^2 x = -\beta x^3$

WE TRY $x = A\cos\omega_0 t + \beta x_1 + \ldots$

PLUGGING IN: $\beta(\ddot{x}_1 + \omega_0^2 x_1) = -\beta(A^3\cos^3\omega_0 t + \ldots)$

OR $\ddot{x}_1 + \omega_0^2 x_1 = -\frac{A^3}{4}(3\cos\omega_0 t + \cos 3\omega_0 t)$

NOW EVEN IN THE 1ST APPROX. WE FIND A DRIVING TERM $\sim \cos\omega_0 t$

$\Rightarrow$ RESONANCE $\Rightarrow$ HUGE AMPLITUDE $\Rightarrow$ CONTRADICTION.

OUR TRIAL SOLUTION $x = A\cos\omega_0 t + \beta x_1$ IS NOT GOOD ENOUGH!

WITHOUT INVOKING DAMPING, WE OBTAIN MORE SATISFACTORY RESULTS IF WE TRY

$$x = A\cos\omega t + \beta x_1 \qquad \text{WITH} \quad \omega \neq \omega_0$$

WE SUPPOSE $\omega = \omega_0 + \beta\omega_1 + \beta^2\omega_2 + \cdots$

LET'S TRY IT. $x = A\cos(\omega_0 + \beta\omega_1)t + \beta x_1$ TO ORDER β

$$\ddot{x} = -A(\omega_0 + \beta\omega_1)^2\cos\omega t + \beta\ddot{x}_1$$

$$\sim -A(\omega_0^2 + 2\beta\omega_0\omega_1)\cos\omega t + \beta\ddot{x}_1 \qquad \text{TO ORDER } \beta$$

PLUGGING INTO $\ddot{x} + \omega_0^2 x = -\beta x^3$ WE FIND

$$-2A\beta\omega_0\cos\omega t + \beta\ddot{x}_1 + \omega_0^2\beta x_1 = -\beta A^3\cos^3\omega t + \cdots$$

NOTING $\omega_0 \sim \omega - \beta\omega_1$, SO $\omega_0^2 \sim \omega^2 - 2\beta\omega\omega_1$ TO ORDER β

AND WE MAY REPLACE $\omega_0^2\beta x_1$ BY $\omega^2\beta x_1$

THEN $\ddot{x}_1 + \omega^2 x_1 = 2A\omega_0\omega_1\cos\omega t - \frac{A^3}{4}(3\cos\omega t + \cos 3\omega t)$

TO AVOID THE RESONANCE, WE CHOOSE ω_1 TO KILL THE $\cos\omega t$ TERMS

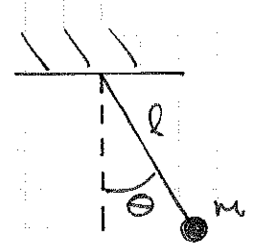
$$\underline{\omega_1 = \frac{3}{8}\frac{A^2}{\omega_0}}$$

AND $\ddot{x}_1 + \omega^2 x_1 = -\frac{A^3}{4}\cos 3\omega t$

$$\Rightarrow x_1 = -\frac{A^3}{4}\frac{\cos 3\omega t}{\omega^2 - 9\omega^2} = \frac{A^3}{32\omega^2}\cos 3\omega t$$

SO OUR SOLUTION IS $x = A\cos\omega t + \frac{\beta A^3}{32\omega^2}\cos 3\omega t$ WHERE $\omega = \omega_0 + \frac{3\beta A^2}{8\omega_0}$

EXAMPLE SIMPLE PENDULUM



$$ml^2\ddot{\theta} = -mgl\sin\theta \sim -mgl\left(\theta - \frac{\theta^3}{6}\right)$$

$$\ddot{\theta} + \omega_0^2\theta = \frac{\omega_0^2}{6}\theta^3 \qquad \left[\omega_0^2 = \frac{g}{l}\right]$$

COMPARING TO OUR FORM ABOVE, $\beta = -\frac{\omega_0^2}{6}$

AND THE APPROXIMATE SOLUTION IS

$$\Theta = \Theta_0 \cos\omega t - \frac{\omega_0^2 \Theta_0^3}{192\,\omega^2} \cos 3\omega t$$

AND $\omega = \omega_0 \left(1 - \frac{\Theta_0^2}{16}\right)$

NOTE $\Theta_{MAX} = \Theta_0 \left(1 - \frac{\omega_0^2 \Theta_0^2}{192\omega^2}\right) \sim \Theta_0$

AND THE PERIOD IS $T = \frac{2\pi}{\omega_0} \frac{1}{1 - \Theta_0^2/16} \sim 2\pi \sqrt{\frac{l}{g}} \left(1 + \frac{\Theta_{MAX}^2}{16} + \ldots\right)$

AS BEFORE.

WE HAVE FOUND 3 FEATURES OF NON-LINEAR OSCILLATIONS FOR THE CASE OF NEARLY LINEAR FORCES:

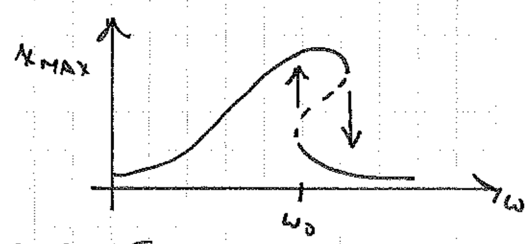
1) DISPLACEMENT OF THE AVERAGE $\langle x \rangle$ – IF THE POTENTIAL IS ASYMMETRIC

2) APPEARANCE OF HIGHER HARMONICS OF THE NATURAL FREQUENCY OF THE LINEAR OSCILLATIONS.

3) SHIFT OF THE PRINCIPAL (= NATURAL) FREQUENCY $\Rightarrow$ CHANGE IN PERIOD.

SEVERAL ADDITIONAL STRIKING FEATURES OCCUR IN FORCED NON-LINEAR OSCILLATIONS. (L & L SEC 29).

4) AMPLITUDE JUMPS FOR LARGE DRIVING FORCES NEAR RESONANCE, THE AMPLITUDE OF OSCILLATION MAY NOT BE SINGLE VALUED.



IF THE DRIVING FREQUENCY IS VARIED, THE AMPLITUDE MAY 'JUMP' DISCONTINUOUSLY FROM ONE BRANCH TO THE OTHER OF THE RESPONSE CURVE. PROCESSES IN NATURE SUCH AS MELTING, BOILING AND OTHER 'PHASE TRANSITIONS' ARE THOUGHT OF AS A KIND OF NON-LINEAR RESPONSE TO SOME DRIVING MECHANISM.

5) SUB-HARMONICS IN SOME CASES IT IS POSSIBLE TO EXCITE RESONANCE IN A NON LINEAR OSCILLATOR WHEN THE DRIVING FREQUENCY IS A SUBHARMONIC OF THE NATURAL FREQUENCY. IE, WHEN $\omega = \omega_0/n$.

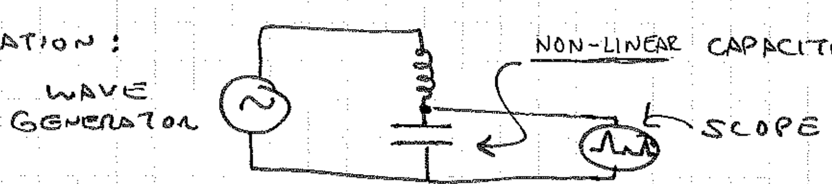
CONVERSELY, A NON LINEAR OSCILLATOR DRIVEN AT RESONANT FREQUENCY ω_0, MAY HAVE A RESPONSE CONTAINING SUBHARMONIC OSCILLATIONS ....

6) FEIGENBAUM CASCADES (SEE SCI AM. NOV 1981, MATH. GAMES)

A LONGSTANDING PROBLEM IN NON-LINEAR SYSTEMS IS HOW TO DESCRIBE THE TRANSITION FROM OSCILLATORY BEHAVIOR TO TURBULENT BEHAVIOR - AS IN THE SWIRLS OF AIR BEHIND A FAST-MOVING CAR, ETC.

FEIGENBAUM HAS RECENTLY NOTED A MATHEMATICAL FACT THAT SOME NON-LINEAR SYSTEMS EXCITE HIGHER AND HIGHER HARMONICS IN A 'UNIVERSAL' MANNER INVOLVING THE DIMENSIONLESS CONSTANTS 2.509... AND 4.669.... THIS SUGGESTS THERE MAY BE SOME UNIVERSAL FEATURE IN THE APPROACH TO TURBULENCE (= CHAOTIC SITUATION WITH INFINITELY MANY HARMONICS PRESENT).

DEMONSTRATION:



NON-LINEAR CAPACITOR IS ACTUALLY A DIODE - SO IS SHORTED OVER HALF A CYCLE.

THE METHOD OF AVERAGES

ANOTHER CLASS OF NON-LINEAR MOTION IS THAT OF A SIMPLE HARMONIC OSCILLATOR SUBJECT TO DAMPING. IN GENERAL THE DAMPING WILL CAUSE A LOSS OF AMPLITUDE WITH TIME. IF THIS LOSS IS SLOW COMPARED TO THE PERIOD OF OSCILLATION, WE MAY USE THE SO-CALLED METHOD OF AVERAGES.

OUR DIFFERENTIAL EQUATION IS

$$\ddot{x} + \omega_0^2 x = \epsilon f(x, \dot{x})$$

DAMPING FUNCTION

IF $\epsilon = 0$, $x = a \cos \omega_0 t$ OF COURSE

FOR $\epsilon \neq 0$ WE TRY A SOLUTION OF A SIMILAR FORM:

$$\underline{x = a(t) \cos \phi(t)}$$

WHERE $a(t)$ IS SLOWLY VARYING, AND $\phi(t) \sim \omega_0 t$

SO FAR, WE HAVE REPLACED ONE UNKNOWN, $x(t)$, BY TWO UNKNOWNS $a(t)$ AND $\phi(t)$. THE MARVELOUS TRICK IS THAT WE CAN ENFORCE A CONSTRAINT ON a & ϕ THAT THE FORM OF THE VELOCITY $\dot{x}$ SHOULD ALSO LOOK VERY MUCH LIKE THAT WHEN $\epsilon = 0$ $\Longrightarrow$ $\dot{x} = -\omega_0 a \sin \omega_0 t$.

OUR CONSTRAINT IS $\underline{\dot{x} = -\omega_0 a(t) \sin \phi(t)}$

WE NOW HAVE 2 EQUATION IN 2 UNKNOWNS a & ϕ

1) $\dot{x} = \underline{-a\omega_0 \sin\phi} = \frac{dx}{dt} = \frac{d}{dt}(a\cos\phi) = \underline{\dot{a}\cos\phi - a\dot{\phi}\sin\phi}$

2) $\ddot{x} + \omega_0^2 x = \epsilon f \Rightarrow \frac{d}{dt}(\dot{x}) + \omega_0^2 x = \epsilon f$

$$\Rightarrow \underline{-\omega_0\dot{a}\sin\phi - \omega_0 a\dot{\phi}\cos\phi + \omega_0^2 a\cos\phi} = \epsilon f$$

REARRANGING THE 2 EQUATIONS:

$$\dot{a}\cos\phi + a(\omega_0 - \dot{\phi})\sin\phi = 0$$

$$-\dot{a}\sin\phi + a(\omega_0 - \dot{\phi})\cos\phi = \frac{\epsilon f}{\omega_0}$$

$$\Rightarrow \quad \dot{a} = -\frac{\epsilon f}{\omega_0}\sin\phi$$

AND $a(\omega_0 - \dot{\phi}) = \frac{\epsilon f}{\omega_0}\cos\phi$ OR $\dot{\phi} = \omega_0 - \frac{\epsilon f}{a\omega_0}\cos\phi$

IF a IS SLOWLY VARYING OUR FINAL TRICK IS TO SUPPOSE THE EQUATIONS FOR $\dot{a}$ & $\dot{\phi}$ CAN BE APPROXIMATED BY THEIR <u>AVERAGE</u> VALUES OVER A PERIOD OF OSCILLATION, $0 < \phi < 2\pi$.

HENCE

$$\boxed{\begin{aligned} \dot{a} &= -\frac{\epsilon}{2\pi\omega_0}\int_0^{2\pi} f\sin\phi\, d\phi \\ \dot{\phi} &= \omega_0 - \frac{\epsilon}{2\pi a\omega_0}\int_0^{2\pi} f\cos\phi\, d\phi \end{aligned}}$$

<u>EXAMPLE</u> $\ddot{x} + \omega_0^2 x = -\epsilon \dot{x}$

THE WELL-KNOWN CASE OF DAMPING WHICH IS LINEAR IN THE VELOCITY.

$$f = -\dot{x} = a\omega_0 \sin\phi$$

$$\dot{a} = -\frac{\epsilon}{2\pi\omega_0}\int_0^{2\pi} a\omega_0 \sin^2\phi \, d\phi = -\frac{a\epsilon}{2}$$

$$\Rightarrow a(t) = a_0 e^{-\frac{\epsilon t}{2}}$$

$$\dot{\phi} = \omega_0 - \frac{\epsilon}{2\pi a\omega_0}\int_0^{2\pi} a\omega_0 \sin\phi\cos\phi \, d\phi = \omega_0$$

$$\Rightarrow \phi = \omega_0 t$$

AND $\underline{x = a_0 e^{-\frac{\epsilon t}{2}} \cos\omega_0 t}$

WHICH IS JUST OUR PREVIOUS SOLUTION FOR <u>SMALL</u> DAMPING FOR WHICH $\omega = \sqrt{\omega_0^2 - \frac{\epsilon^2}{4}} \sim \omega_0$ $\left(\text{i.e. } \beta = \frac{\epsilon}{2}\right)$

<u>EXAMPLE</u> THE SIMPLE PENDULUM

$$\ddot{\theta} + \omega_0^2\theta = \frac{\omega_0^2\theta^3}{6} \qquad \omega_0^2 = g/l$$

HENCE $\epsilon f = \frac{\omega_0^2\theta^3}{6} = \frac{\omega_0^2}{6}a^3\cos^3\phi$ USING $\theta = a\cos\phi$

$$\dot{a} = -\frac{\omega_0 a^3}{12\pi}\int_0^{2\pi}\cos^3\phi\sin\phi \, d\phi = 0$$

$$\dot{\phi} = \omega_0 - \frac{\omega_0 a^2}{12\pi}\int_0^{2\pi}\cos^4\phi \, d\phi = \omega_0\left(1 - \frac{a^2}{16}\right)$$

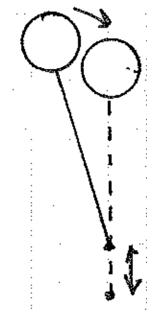
SO $\theta = \theta_{MAX}\cos\left[\omega_0\left(1 - \frac{\theta_{MAX}^2}{16}\right)t\right]$ AS BEFORE!

THIS METHOD DOES NOT PREDICT THE HIGHER HARMONICS, HOWEVER.

Ph 205 Lecture 15

Pendulum with Rapidly Oscillating Support (L&L sec 30)

[For a rather different discussion, see H.P. Kalmus, Am. J. Phys 38, 874 (1970)]

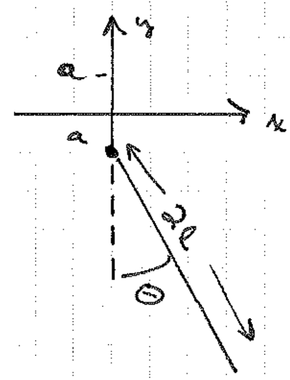


An inverted pendulum can be stable if the point of support oscillates vertically with high frequency. As the support moves downward, the bob is pulled towards the vertical — a kind of restoring action. This might cause stability.

We wish to demonstrate that such an inverted equilibrium actually exists.

First we derive the equation of motion for a pendulum with oscillating support (L&L p11, B&O sec 2-10)

[Can you do it by elementary methods?]



We consider the case of a rod of length 2ℓ, mass m (no bob as such). The support moves according to

$$y = a\cos\omega t$$

The C.M. has coords $x = \ell\sin\theta$, $y = a\cos\omega t - \ell\cos\theta$

so $\dot{x} = \ell\cos\theta\,\dot{\theta}$ $\quad \dot{y} = -a\omega\sin\omega t + \ell\sin\theta\,\dot{\theta}$

$L = T - V \qquad T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 \qquad V = mgy \qquad I = \frac{1}{3}m\ell^2$

$$L = \frac{m}{2}\left(\frac{4}{3}\ell^2\dot{\theta}^2 - 2a\omega\ell\sin\omega t\sin\theta\,\dot{\theta} + a^2\omega^2\sin^2\omega t\right) - mg\left(a\cos\omega t - \ell\cos\theta\right)$$

The equation of motion is

$$\underline{\frac{4}{3}\ddot{\theta} + \frac{g}{\ell}\sin\theta - \frac{a\omega^2}{\ell}\cos\omega t\sin\theta = 0}$$

Vertical oscillation of support

If the support oscillates horizontally, $x_{cm} = a\cos\omega t + \ell\sin\theta$

$y_{cm} = -\ell\cos\theta$

leading to

$$\underline{\frac{4}{3}\ddot{\theta} + \frac{g}{\ell}\sin\theta - \frac{a\omega^2}{\ell}\cos\omega t\cos\theta = 0}$$

Horizontal oscillation of support

THESE EQUATIONS HAVE THE FORM

$$C\ddot{\theta} = F_1(\theta) + F_2(\theta, t)$$

WHERE F_2 IS RAPIDLY OSCILLATING: $F_2 \sim \cos\omega t$ OR $\sin\omega t$

WE TRY A VARIATION ON THE METHOD OF AVERAGES, AND LOOK FOR A SOLUTION LIKE

$$\theta = \theta_1(t) + \theta_2(t)$$

WHERE θ_1 IS SLOWLY VARYING, AND θ_2 IS SMALL AND RAPIDLY OSCILLATING ABOUT ZERO: $\theta_2 \sim \cos\omega t$ OR $\sin\omega t$.

IF $\theta_1 =$ CONSTANT, THE MOTION IS JUST SMALL OSCILLATIONS ABOUT A STABLE EQUILIBRIUM.

THE METHOD IS TO AVERAGE OVER THE RAPID OSCILLATIONS TO CONSTRUCT AN EFFECTIVE POTENTIAL FOR THE SLOW MOTION θ_1. FROM THE EFFECTIVE POTENTIAL WE CAN EASILY JUDGE THE QUESTION OF STABILITY. (KAPITZA 1951)

PLUGGING OUR TRIAL SOLUTION INTO THE DIFFERENTIAL EQUATION:

$$C(\ddot{\theta}_1 + \ddot{\theta}_2) \approx F_1(\theta_1) + \frac{dF_1}{d\theta}\theta_2 + F_2(\theta_1) + \frac{dF_2}{d\theta}\theta_2$$

IN THIS EQUATION THE RAPIDLY OSCILLATING TERMS MUST SEPARATELY BE EQUAL:

$$C\ddot{\theta}_2 = -\omega^2 C\theta_2 = F_2(\theta_1) + \left(\frac{dF_1}{d\theta} + \frac{dF_2}{d\theta}\right)\theta_2$$

NOW θ_2 IS SMALL, BUT ω^2 IS LARGE (RAPID OSCILLATIONS).

SO WE CAN NEGLECT THE TERMS IN θ_2 ON THE RIGHT, YIELDING

$$\theta_2 \sim -\frac{F_2}{\omega^2 C}$$

THE DIFFERENTIAL EQUATION IS NOW

$$C\ddot{\theta}_1 = F_1 - \frac{F_2}{\omega^2 C}\frac{dF_1}{d\theta} - \frac{F_2}{C\omega^2}\frac{dF_2}{d\theta} = F_1 - \frac{F_2}{C\omega^2}\frac{dF_1}{d\theta} - \frac{1}{2C\omega^2}\frac{dF_2^2}{d\theta}$$

IN THE SPIRIT OF THE METHOD OF AVERAGES, WE SUPPOSE THE MOTION $\Theta_1(t)$ IS SUFFICIENTLY WELL DETERMINED IF WE AVERAGE THIS EQUATION OVER A PERIOD OF THE RAPID OSCILLATIONS.

$$C\ddot{\Theta}_1 \sim F_1 - \frac{1}{2C\omega^2}\left\langle \frac{dF_2^2}{d\Theta} \right\rangle = F_1 - \frac{1}{2C\omega^2}\frac{d\langle F_2^2\rangle}{d\Theta}$$

IF $F_1(\Theta) = -\frac{dV}{d\Theta}$, V = POTENTIAL ENERGY

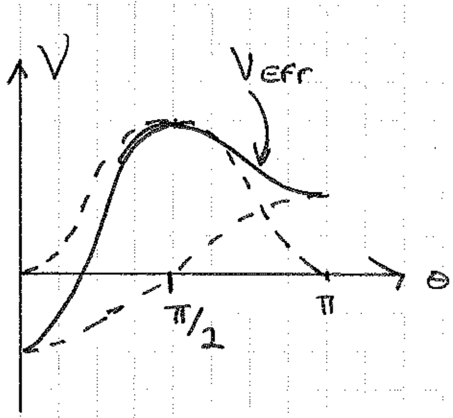
THEN $C\ddot{\Theta}_1 = -\frac{dV_{eff}}{d\Theta}$

WHERE $\boxed{V_{eff} = V + \frac{\langle F_2^2\rangle}{2C\omega^2}}$

IF V_{eff} HAS A MINIMUM THEN WE CAN HAVE STABLE OSCILLATIONS ABOUT THAT POINT, PROVIDED $V''_{eff}(\theta_{min}) > 0$.

CASE 1. VERTICAL OSCILLATION OF SUPPORT

$$\underset{\uparrow \atop C}{\frac{4}{3}}\ddot{\Theta} = \underbrace{-\frac{g}{l}\sin\Theta}_{F_1} + \underbrace{\frac{a\omega^2}{l}\cos\omega t\sin\Theta}_{F_2}$$



$V = -\frac{g}{l}\cos\Theta \qquad \langle F_2^2\rangle = \frac{1}{2}\frac{a^2\omega^4}{l^2}\sin^2\Theta$

SO $V_{eff} = -\frac{g}{l}\cos\Theta + \frac{3}{16}\frac{a^2\omega^2}{l^2}\sin^2\Theta$

$$\frac{dV_{eff}}{d\Theta} = \frac{g}{l}\sin\Theta + \frac{3}{8}\frac{a^2\omega^2}{l^2}\sin\Theta\cos\Theta$$

$$\frac{d^2V_{eff}}{d\Theta^2} = \frac{g}{l}\cos\Theta + \frac{3}{8}\frac{a^2\omega^2}{l^2}(2\cos^2\Theta - 1)$$

$\frac{dV_{eff}}{d\Theta} = 0$ AT $\Theta = 0, \pi$ AND $\cos\Theta = -\frac{8}{3}\frac{gl}{a^2\omega^2}$

THE EQUILIBRIUM AT $\Theta = 0$ IS ALWAYS STABLE

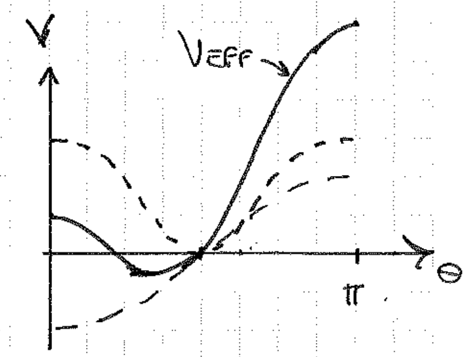
$\cos\Theta = -\frac{8}{3}\frac{g\ell}{a^2\omega^2}$ IS ALWAYS UNSTABLE

AND $\Theta = \pi$ IS STABLE IF $\underline{\omega^2 > \frac{8}{3}\frac{g\ell}{a^2}}$

FOR $a = .75$ cm, $2\ell = 22$ cm, WE NEED $\omega \gtrsim 225 \Rightarrow f > 36$ Hz

WHICH IS NOT ALL THAT RAPID!

CASE 2 HORIZONTAL OSCILLATION OF SUPPORT



$$V_{EFF} = -\frac{g}{\ell}\cos\Theta + \frac{3}{16}\frac{a^2\omega^2}{\ell^2}\cos^2\Theta$$

$$\frac{dV_{EFF}}{d\Theta} = \frac{g}{\ell}\sin\Theta - \frac{3}{8}\frac{a^2\omega^2}{\ell^2}\cos\Theta\sin\Theta$$

$$\frac{d^2V_{EFF}}{d\Theta^2} = \frac{g}{\ell}\cos\Theta - \frac{3}{8}\frac{a^2\omega^2}{\ell^2}(2\cos^2\Theta - 1)$$

THE EQUILIBRIUM AT $\Theta = 0$ IF STABLE IF $\omega^2 < \frac{8}{3}\frac{g\ell}{a^2}$

$\cos\Theta = \frac{8}{3}\frac{g\ell}{a^2\omega^2}$ IS STABLE IF $\omega^2 > \frac{8}{3}\frac{g\ell}{a^2}$

$\Theta = \pi$ IS ALWAYS UNSTABLE

IN THE LIMIT $\omega \to \infty$ THE MOTION IS STABLE FOR THE ROD ALIGNED ALONG THE DIRECTION OF OSCILLATION OF SUPPORT — WHATEVER IT MAY BE.