

STANDING WAVES, CONTINUED

ENERGY CONSIDERATIONS

WE USE THE TRANSVERSE VIBRATIONS ON A STRING AS AN EXAMPLE. THE DISPLACEMENT IS  $S(x,t)$ .

$$K.E. = I = \int_0^l \frac{1}{2} \rho v^2 dx = \frac{\rho}{2} \int_0^l \dot{s}^2 dx \quad (\text{IF } \rho \text{ CONSTANT})$$

$$P.E. = \text{WORK DONE IN STRETCHING THE STRING} = \int T dl$$

(T = TENSION, I = KINETIC ENERGY)

$$\text{SO } V = T \left\{ \int_0^l \sqrt{1 + \left(\frac{ds}{dx}\right)^2} dx - l \right\} \approx \frac{T}{2} \int_0^l s'^2 dx$$

AGAIN WE IGNORE THE POSSIBILITY THAT THE TENSION MIGHT CHANGE DUE TO THE STRETCH.

EXAMPLE THE PLUCKED STRING (p. 228)

$$S(x,t) = \sum_n A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$\dot{s} = - \sum_n A_n \frac{n\pi c}{l} \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}$$

$$s' = \sum_n A_n \frac{n\pi}{l} \cos \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$I = \frac{\rho}{2} \int_0^l dx \sum_{m,n} A_m A_n \left(\frac{n\pi c}{l}\right) \left(\frac{m\pi c}{l}\right) \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} \sin \frac{m\pi ct}{l} \sin \frac{n\pi ct}{l}$$

BUT ALL INTEGRALS VANISH UNLESS  $n=m$ , FOR WHICH THEY ARE  $l/2$ .

$$I = \frac{\rho}{2} \frac{l}{2} \left(\frac{\pi c}{l}\right)^2 \sum_n n^2 A_n^2 \sin^2 \frac{n\pi ct}{l} \Rightarrow \langle I \rangle_{\text{TIME}} = \frac{\rho \pi^2 c^2}{8l} \sum_n n^2 A_n^2$$

$$\text{LIKEWISE } V = \frac{T}{2} \frac{l}{2} \left(\frac{\pi}{l}\right)^2 \sum_n n^2 A_n^2 \cos^2 \frac{n\pi ct}{l} \Rightarrow \langle V \rangle_{\text{TIME}} = \frac{T \pi^2}{8l} \sum_n n^2 A_n^2$$

RECALL THAT  $c^2 = T/\rho$ , SO

$$I + V = \frac{T \pi^2}{4l} \sum_n n^2 A_n^2 = \text{CONSTANT} \quad \text{AND } \langle I \rangle = \langle V \rangle \quad (\text{VIRIAL THEOREM!})$$

APPROXIMATION TO THE LOWEST FREQUENCY

RAYLEIGH HAS GIVEN A VERY CLEVER METHOD OF APPROXIMATION TO THE LOWEST FREQUENCY OF VIBRATION, USING AN ENERGY METHOD.

WE GUESS AT A FORM  $S(x,t) = f(x,p) \cos \omega t$ , WHERE  $f$  IS A FAMILY OF CURVES DEPENDENT ON PARAMETER  $p$ .  $f$  MUST SATISFY THE BOUNDARY CONDITIONS, OF COURSE.

WE NOW REQUIRE  $\langle I \rangle = \langle V \rangle$

$$\text{HENCE } \frac{\rho \omega^2}{4} \int f^2 dx = \frac{T}{4} \int f'^2 dx$$

WHICH RELATES  $\omega^2$  TO THE PARAMETER  $P$ .

THE INSIGHT OF RAYLEIGH IS THAT LOWEST IS LOWEST! SO WE VARY  $P$  TO MINIMIZE  $\omega$ . THE ACCURACY OBTAINABLE BY THIS PROCEDURE IS REMARKABLE. BY CHOOSING  $f$  TO BE A POLYNOMIAL, ALL INTEGRALS ARE ELEMENTARY....

WE LEAVE EXAMPLES TO YOU ON THE PROBLEM SET.

### APPROXIMATE SOLUTION FOR VARIABLE DENSITY STRINGS

THE CASE OF A STRING WITH ARBITRARILY VARYING DENSITY IS QUITE DIFFICULT. BUT IF THE DENSITY IS ALMOST UNIFORM WE MAY USE  $\langle I \rangle = \langle V \rangle$  TO OBTAIN AN APPROXIMATE SOLUTION — AGAIN DUE TO RAYLEIGH.

$$\text{SUPPOSE } \rho(x) = \rho_0 + \rho_1(x)$$

WHERE  $\rho_0 = \text{CONSTANT}$  AND  $\rho_1$  IS SMALL

WE CONSIDER THE NORMAL MODES. FOR A STRING FIXED AT BOTH ENDS, WE SUPPOSE THESE CAN BE WRITTEN

$$S_n(x, t) = \sin \frac{n\pi x}{l} \cos \Omega_n t$$

RECALL THAT IF  $\rho_1 = 0$  AND  $c^2 = T/\rho_0$  THEN  $\omega_n = \frac{n\pi c}{l}$

$$\langle I_n \rangle = \frac{\Omega_n^2}{4} \int_0^l \rho(x) \sin^2 \frac{n\pi x}{l} dx = \Omega_n^2 \left( \frac{\rho_0 l}{8} + \frac{1}{4} \int_0^l \rho_1(x) \sin^2 \frac{n\pi x}{l} dx \right)$$

$$\langle V_n \rangle = \frac{T}{4} \int_0^l \left( \frac{n\pi}{l} \right)^2 \cos^2 \frac{n\pi x}{l} dx = \frac{Tl}{8} \left( \frac{n\pi}{l} \right)^2 = \frac{\rho_0 l}{8} \left( \frac{n\pi c}{l} \right)^2 = \frac{\rho_0 l}{8} \omega_n^2$$

$$\text{HENCE } \Omega_n^2 = \omega_n^2 / \left( 1 + \frac{2}{\rho_0 l} \int_0^l \rho_1(x) \sin^2 \frac{n\pi x}{l} dx \right)$$

$$\text{OR } \boxed{\Omega_n \approx \omega_n \left( 1 - \frac{1}{\rho_0 l} \int_0^l \rho_1(x) \sin^2 \frac{n\pi x}{l} dx \right)}$$

THIS RESULT, DERIVED QUICKLY BY A PHYSICS 'TRICK', IS AN EXAMPLE OF THE RESULTS OF PERTURBATION THEORY. THIS TECHNIQUE WILL BE VERY PROMINENT IN THE QUANTUM MECHANICS OF COMPLICATED SYSTEMS.

AGAIN WE LEAVE EXAMPLES TO THE PROBLEM SET.

BUT NOTE THAT IF  $\rho$  IS POSITIVE THEN  $\omega_n < \omega_n$ . IF THE MASS OF THE STRING IS INCREASED IN ANY MANNER, THEN THE FREQUENCY OF VIBRATION DECREASES!

### LAGRANGE'S EQUATIONS OF MOTION

LAGRANGE HAS GIVEN AN ALTERNATIVE TO  $F=ma$ , WHICH CAN BE APPLIED TO CONTINUOUS SYSTEMS ALSO.

$$L = T - V = \int_0^l \left( \frac{\rho \dot{s}^2}{2} - \frac{T s'^2}{2} \right) dx$$

WE WRITE  $L = \int \mathcal{L}(s, \dot{s}, s', x, t) dx$

WHERE  $\mathcal{L} = \frac{\rho \dot{s}^2}{2} - \frac{T s'^2}{2} = \underline{\text{LAGRANGIAN DENSITY}}$

NOTE THAT  $x$  DOES NOT PLAY THE ROLE OF A GENERALISED COORDINATE IN  $\mathcal{L}$ , BUT RATHER IS A PARAMETER LIKE  $t$ . INSTEAD,  $s$  PLAYS THE ROLE OF THE GENERALISED COORDINATE. SINCE  $s(x)$  IS A CONTINUOUS FUNCTION, WE HAVE AN INFINITE NUMBER OF GENERALISED COORDINATES.

TO OBTAIN THE EQUATIONS OF MOTION, IT IS QUICKEST TO USE HAMILTON'S PRINCIPLE:

$$\delta \int_{t_1}^{t_2} L dt = 0, \text{ FOR}$$

VARIATIONS ABOUT THE TRUE MOTION.

$$\text{NOW } \delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} \int_0^l \mathcal{L}(s, \dot{s}, s', x, t) dx dt = 0$$

RECALL THE METHOD OF THE CALCULUS OF VARIATIONS:

$$\text{LET } s(x, t) \rightarrow s(x, t) + \epsilon \eta(x, t)$$

$$\text{THEN } \dot{s} \rightarrow \dot{s} + \epsilon \dot{\eta}, \quad s' \rightarrow s' + \epsilon \eta'$$

$$\text{AND } \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial s} \epsilon \eta + \frac{\partial \mathcal{L}}{\partial \dot{s}} \epsilon \dot{\eta} + \frac{\partial \mathcal{L}}{\partial s'} \epsilon \eta' \quad (\text{WE DON'T VARY TIME})$$

INTEGRATE BY PARTS:

$$\delta \int L dt = \epsilon \iint \left[ \frac{\partial \mathcal{L}}{\partial s} \eta - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{s}} \right) \eta - \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial s'} \right) \eta \right] dx dt + \eta \frac{\partial \mathcal{L}}{\partial \dot{s}} \Big|_{t_1}^{t_2} + \eta \frac{\partial \mathcal{L}}{\partial s'} \Big|_0^l$$

$$\text{NOW } \eta \frac{\partial \mathcal{L}}{\partial \dot{s}} \Big|_{t_1}^{t_2} = 0 \text{ BY DEFINITION OF } \eta.$$

WE GET AN UNEXPECTED PAYOFF BY REQUIRING  $\eta \frac{\partial \mathcal{L}}{\partial s'} \Big|_{0 \text{ or } l} = 0$

THIS WILL GIVE US THE BOUNDARY CONDITIONS!

IF THE ENDS OF THE STRING ARE FIXED,  $s(0,t) = 0 = s(l,t)$   
AND SO  $\eta(0,t) = 0 = \eta(l,t)$  ALSO.

BUT IF THE ENDS ARE FREE, WE MUST HAVE  $\frac{\partial \mathcal{L}}{\partial s'} \Big|_{0 \text{ or } l} = 0$

$\Rightarrow s'(0,t) = 0 = s'(l,t)$  - WHICH WE DERIVED BEFORE BY ANOTHER ARGUMENT.

RETURNING TO  $\delta \int L dt = 0$ , SINCE  $\epsilon \eta$  IS ARBITRARY,

WE MUST HAVE  $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{s}} + \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial s'} - \frac{\partial \mathcal{L}}{\partial s} = 0$

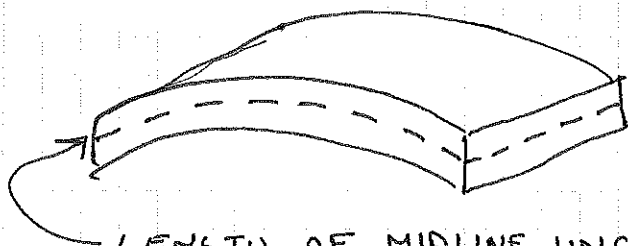
WHEN  $\mathcal{L} = \frac{\rho \dot{s}^2}{2} - \frac{T s'^2}{2}$ , WE GET  $\rho \ddot{s} - T s'' = 0$  AS EXPECTED!

### TRANSVERSE VIBRATIONS OF AN ELASTIC BAR OR SPRING

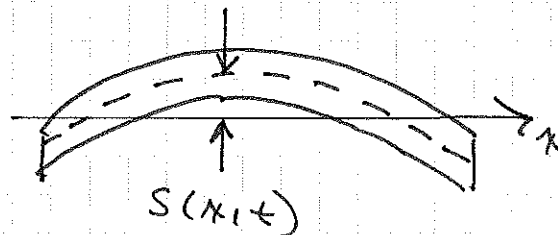
WE NOW EXAMINE A MORE COMPLICATED PROBLEM - TRANSVERSE VIBRATIONS OF A BAR OF FINITE CROSS-SECTIONAL AREA. A PREVIEW OF THIS PROBLEM IN THE STATIC CASE WAS GIVEN IN PROBLEM (6), SET 3.

ALTHOUGH THE EQUATION OF MOTION CAN BE GOTTEN FROM  $F=ma$ , WE WILL USE LAGRANGE'S METHOD. THIS WILL ALSO TELL US THE BOUNDARY CONDITIONS - WHICH WAS A STICKY POINT HISTORICALLY FOR THIS PROBLEM.

WE MAKE A SIMPLIFYING ASSUMPTION THAT THE MIDPLANE OF THE BAR IS NEITHER STRETCHED NOR COMPRESSED.



LENGTH OF MIDLINE UNCHANGED

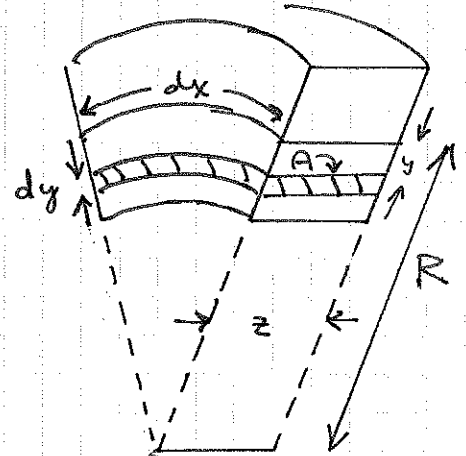


WE CALL THE DISPLACEMENT OF THE MIDPLANE  $s(x,t)$ .

DISPLACEMENTS OF PARTICLES WHICH ARE NOT IN THE MIDPLANE ARE RELATED TO  $s(x,t)$  BY SUPPOSING THAT TRANSVERSE SECTIONS THRU THE BAR ALWAYS REMAIN PLANES - BUT THEY CAN ROTATE ABOUT THEIR LINE OF INTERSECTION WITH THE MIDPLANE.

WE IGNORE STRETCHING AND COMPRESSION IN THE TRANSVERSE DIRECTION.

FIRST WE CALCULATE THE POTENTIAL ENERGY STORED UPON DEFORMING A SECTION OF THE BAR INTO A WEDGE. THE SECTION IS  $dx$  LONG, AND ALONG THE DEFORMED MIDPLANE  $dx$  REMAINS CONSTANT.



RECALL THAT  $\frac{\text{FORCE}}{\text{AREA}} = Y \frac{\Delta l}{l}$  FOR AN ELASTIC SOLID, WHERE  $Y = \text{YOUNG'S MODULUS}$ .

THEN THE FORCE NEEDED TO COMPRESS OR STRETCH A SUBSECTION IS

$$F = A Y \frac{\Delta l}{l} \quad \text{AND HENCE} \quad d.P.E. \equiv dV = \frac{1}{2} \left( \frac{AY}{l} \right) \Delta l^2$$

length of the subsection

FOR A SUBSECTION OF THE BAR AT DISTANCE  $y$  FROM THE MIDPLANE,

$$A = z(y) dy, \quad l = dx \quad \text{AND} \quad \Delta l = dx \frac{R+y}{R} - dx = \frac{y dx}{R}$$

WHERE  $R = \text{RADIUS OF CURVATURE OF THE MIDPLANE}$ .

$$\text{SO } dV = \frac{1}{2} Y \frac{z(y) dy}{dx} \frac{y^2 dx^2}{R^2} = \frac{Y}{2R^2} z y^2 dy dx$$

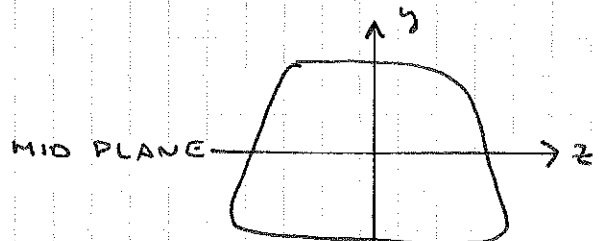
FIRST WE INTEGRATE OVER  $y$ :

$$\int z(y) y^2 dy = I/\rho$$

WHERE  $\rho = \text{VOLUME DENSITY}$

AND  $I = \text{MOMENT OF INERTIA PER UNIT LENGTH ALONG THE BAR}$

$$\text{HENCE} \quad dV = \frac{Y I}{2 \rho R^2} dx$$



AS A SIDE LIGHT, WE NOTE THAT THE STIFFNESS OF A BEAM IS PROPORTIONAL TO ITS MOMENT OF INERTIA PER UNIT LENGTH.

HENCE THE CLASSIC I-BEAM



IS STRONG BUT LIGHT.

PH 205 LECTURE 22

THE RADIUS OF CURVATURE CAN BE WRITTEN

$$\frac{1}{R} = \frac{d\theta}{dl}$$

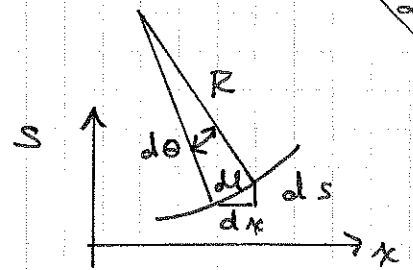
FOR SMALL DISPLACEMENTS,

$$\theta \sim \tan \theta = ds/dx$$

WHILE  $dl = \sqrt{1 + \left(\frac{ds}{dx}\right)^2} dx \sim dx$

SO  $\frac{1}{R} \sim \frac{d}{dx} \left(\frac{ds}{dx}\right) = \frac{d^2s}{dx^2} = s''$

ALTOGETHER  $V = \frac{YI}{2\rho} \int (s'')^2 dx$



THE KINETIC ENERGY HAS TWO PARTS, TRANSLATION AND ROTATION:

TRANSLATION:  $T_T = \frac{1}{2} \int \rho A dx \dot{s}^2$       A = CROSS-SECTIONAL AREA

ROTATION:  $T_R = \frac{1}{2} \int I dx \left(\frac{d\theta}{dt}\right)^2$

AGAIN  $\theta \sim ds/dx$ , SO  $T_R \sim \frac{I}{2} \int \left(\frac{d^2s}{dt dx}\right)^2 dx = \frac{I}{2} \int (\dot{s}')^2 dx$

ALTOGETHER  $L = T - V = \frac{1}{2} \int dx \left[ \rho A \dot{s}^2 + I (\dot{s}')^2 - \frac{YI}{\rho} (s'')^2 \right]$

$L = \int \mathcal{L}(\dot{s}, \dot{s}', s'') dx$

AGAIN HAMILTON'S PRINCIPLE IS THAT  $\delta \int L dt = 0$ , FOR VARIATIONS  $s \rightarrow s + \eta$

$$\delta \int L dt = \iint \left[ \frac{\partial \mathcal{L}}{\partial \dot{s}} \dot{\eta} + \frac{\partial \mathcal{L}}{\partial \dot{s}'} \dot{\eta}' + \frac{\partial \mathcal{L}}{\partial s''} \eta'' \right] dx dt = 0$$

INTEGRATE BY PARTS:

$$0 = \iint \left[ -\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{s}} \eta - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \dot{s}'} \eta' - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial s''} \eta'' \right] dx dt + \eta \frac{\partial \mathcal{L}}{\partial \dot{s}} \Big|_{t_1}^{t_2} + \eta' \frac{\partial \mathcal{L}}{\partial \dot{s}'} \Big|_{x_1}^{x_2} + \eta'' \frac{\partial \mathcal{L}}{\partial s''} \Big|_{x_1}^{x_2}$$

INTEGRATE BY PARTS AGAIN:

$$0 = \iint \eta \left[ -\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{s}} + \frac{d^2}{dx dt} \frac{\partial \mathcal{L}}{\partial \dot{s}'} + \frac{d^2}{dx^2} \frac{\partial \mathcal{L}}{\partial s''} \right] dx dt$$

$$+ \eta \frac{\partial \mathcal{L}}{\partial \dot{s}} \Big|_{t_1}^{t_2} + \eta' \frac{\partial \mathcal{L}}{\partial \dot{s}'} \Big|_{t_1}^{t_2} + \eta' \frac{\partial \mathcal{L}}{\partial s''} \Big|_{x_1}^{x_2} - \eta \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \dot{s}'} \Big|_{x_1}^{x_2} - \eta \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial s''} \Big|_{x_1}^{x_2}$$

WE SET THE CONSTANT TERMS TO ZERO TO PROVIDE THE BOUNDARY CONDITIONS. THE EQUATION OF MOTION FOLLOWS FROM THE INTEGRAL, SINCE  $\eta$  IS ARBITRARY:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{s}} - \frac{d^2}{dx dt} \frac{\partial \mathcal{L}}{\partial \dot{s}'} - \frac{d^2}{dx^2} \frac{\partial \mathcal{L}}{\partial s''} = 0$$

$$\text{OR } \underline{PA \ddot{s} - I \ddot{s}'' + \frac{\gamma I}{\rho} s'''' = 0}$$

THIS IS A 4TH ORDER PARTIAL DIFFERENTIAL EQUATION!

FOR SMALL DISPLACEMENTS,  $s''$  IS SMALL, AND WE CAN NEGLECT THE ROTARY KINETIC ENERGY TERM  $I \ddot{s}''$  RELATIVE TO THE TRANSLATIONAL TERM  $PA \ddot{s}$ . (WE CANNOT NEGLECT THE TERM  $\frac{\gamma I}{\rho} s''''$  OR WE THROW AWAY THE FORCE!  $\gamma$  IS BIG IN PRACTICE.)

WE RESTRICT OURSELVES TO THE APPROXIMATE EQUATION

$$PA \ddot{s} + \frac{\gamma I}{\rho} s'''' = 0 \quad \text{OR} \quad \ddot{s} + \frac{\gamma I}{\rho^2 A} s'''' = 0$$

RECALL THAT  $I$  = MOMENT OF INERTIA PER UNIT LENGTH ALONG BAR

$PA$  = MASS PER UNIT LENGTH ALONG BAR.

$\Rightarrow \frac{I}{PA}$  IS AN AREA. CALL IT  $Ad^2$  [d IS DIMENSIONLESS]

FOR A SQUARE BAR,  $d^2 = 1/12$

FOR A CIRCULAR BAR  $d^2 = 1/4\pi$

WE ALSO DEFINE  $c^2 \equiv \frac{\gamma \rho}{\rho}$  [c HAS DIMENSIONS  $\frac{\text{LENGTH}^2}{\text{TIME}}$ ]

IN THIS NOTATION, OUR WAVE EQUATION IS

$$\ddot{s} + (cd)^2 s'''' = 0$$

BOUNDARY CONDITIONS

THE BOUNDARY CONDITIONS FOR TRANSVERSE VIBRATIONS OF A BAR ARE MORE COMPLICATED THAN FOR LONGITUDINAL VIBRATIONS.

WE CONSIDER ONLY THE 2 COMMON CASES:

1) CLAMPED END. CERTAINLY  $s(x_0, t) = 0$  [ $x_0 =$  POINT OF CLAMP]

BUT LOOKING AT THE TERMS ON P 241, WE SEE WE MUST ALSO REQUIRE  $\eta'(x_0, t) = 0$ , AND CORRESPONDINGLY  $s'(x_0, t) = 0$

PHYSICALLY, IF YOU CLAMP A BAR, YOU CONSTRAIN ITS SLOPE AS WELL AS ITS POSITION.

2) FREE END NOW WE CANNOT EXPECT  $\eta$  OR  $\eta'$  TO VANISH AT  $x_0$ . INSTEAD WE MUST HAVE  $\frac{\partial \tau}{\partial s''} = 0 = \frac{d}{dx} \frac{\partial \tau}{\partial s''}$ , (AGAIN REFERRING TO P. 241).

$$\Rightarrow s''(x_0, t) = 0 = s'''(x_0, t)$$

PHYSICALLY, THE B.M. IS STRAIGHT AT THE FREE END, EVEN THOUGH IT MAY BE DEFLECTED.

THE CONDITION  $s''' = 0$  MIGHT EASILY BE MISSED WITHOUT LAGRANGE'S HELP.

EXAMPLE BOTH ENDS FREE

$$\ddot{s} + c^2 d^2 s'''' = 0 \quad \& \quad s''(0, t) = s'''(0, t) = s''(l, t) = s'''(l, t)$$

WE LOOK FOR AN OSCILLATORY SOLUTION:  $s = f(x) \cos \omega t$

$$\Rightarrow s'''' = \frac{\omega^2}{c^2 d^2} s$$

THIS CAN BE SATISFIED BY 4 KINDS OF FUNCTIONS

$\cos kx$ ,  $\sin kx$ ,  $\cosh kx$ ,  $\sinh kx$

(4TH ORDER  $\Rightarrow$  4 SOLUTIONS)

WHERE  $k = \sqrt{\frac{\omega}{cd}}$  OR  $\omega = cdk^2$



Then  $f(x) = A \cos kx + B \sin kx + C \cosh kx + D \sinh kx$   
 $f' = -Ak \sin kx + Bk \cos kx + Ck \sinh kx + Dk \cosh kx$   
 $f'' = -Ak^2 \cos kx - Bk^2 \sin kx + Ck^2 \cosh kx + Dk^2 \sinh kx$   
 $f''' = Ak^3 \sin kx - Bk^3 \cos kx + Ck^3 \sinh kx + Dk^3 \cosh kx$

AND  $f''(0) = 0 \Rightarrow A = C$

$f'''(0) = 0 \Rightarrow B = D$

$f''(l) = 0 \Rightarrow A(\cosh kl - \cos kl) = B(\sin kl - \sinh kl)$

$f'''(l) = 0 \Rightarrow A(\sinh kl + \sin kl) = B(\cos kl - \cosh kl)$

UNFORTUNATELY WE CANNOT SET ONE OF A OR B TO ZERO

AS  $\cos kl - \cosh kl \neq 0$  UNLESS  $kl = 0$ .

INSTEAD WE FIND  $(\cosh kl - \cos kl)^2 = \sinh^2 kl - \sin^2 kl$   
 OR  $\cosh^2 kl - 2 \cosh kl \cos kl + \cos^2 kl - \sinh^2 kl + \sin^2 kl = 0$

OR  $\boxed{\cosh kl \cos kl = 1}$

THIS EQUATION HAS MANY ROOTS  $\Rightarrow$  MANY NORMAL MODES.

WHEN  $kl$  IS LARGE,  $\cosh kl \rightarrow \infty \Rightarrow \cos kl \rightarrow 0$

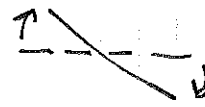
$\Rightarrow kl \approx (2n+1)\pi/2$

WITH YOUR CALCULATOR YOU CAN VERIFY THAT SOME ROOTS ARE

$kl = 4.73004, 7.853205, 10.995608 \dots$

THE ROOT WITH  $n=0$ ,  $kl \approx \pi/2$  IS MISSING

THIS CORRESPONDS TO A MOTION



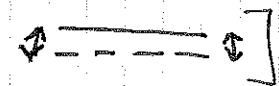
, WHICH IS JUST

A ROTATION  $\Rightarrow$  NO RESTORING FORCE.

THE LOWEST MODE HAS TWO NODES:



[ OF COURSE, THE MODE  $\omega = k = 0$  IS GONE ALSO



BOTH ENDS CLAMPED

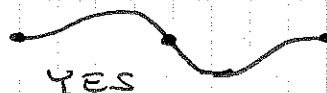
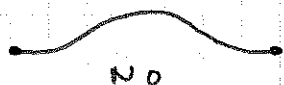
$$f(0) = f'(0) = f(l) = f'(l) = 0$$

WE QUICKLY SEE THAT  $A = -C$ ,  $B = -D$

BUT  $\cosh kL \cos kL = 1$  AGAIN

SO THE FREQUENCIES ARE THE SAME AS FOR BOTH ENDS FREE.

AGAIN  $kL \approx (2n+1)\pi/2$ , AND THE  $n=0$  MODE IS MISSING



ONE END CLAMPED, ONE END FREE

SAY  $x=0$  IS CLAMPED

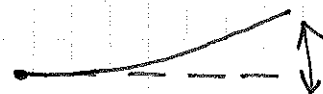
$$\Rightarrow A = -C, B = -D$$

BUT NOW  $\cosh kL \cos kL = -1$

$$kL = 1.8751, 4.69409, 7.85476 \dots$$

AGAIN AS  $kL \rightarrow \infty$ ,  $kL \approx (2n+1)\pi/2$

THE  $n=0$  MODE IS PRESENT NOW



WAVES ON A MEMBRANE (DRUM HEAD)

AS OUR FINAL EXAMPLE OF STANDING WAVES, WE CONSIDER A MEMBRANE, OR SHEET, STRETCHED WITH A UNIFORM SURFACE TENSION  $T$ .

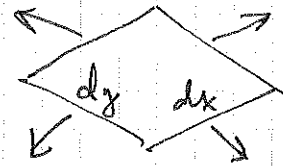
BY SURFACE TENSION, WE MEAN THAT IF A SLIT  $dl$  LONG WERE CUT IN THE MEMBRANE, A FORCE  $T dl$  WOULD BE  $\perp$  TO THE SLIT WOULD BE REQUIRED TO HOLD IT CLOSED. THEN  $T$  IS THE SURFACE TENSION. IT HAS DIMENSIONS OF FORCE/LENGTH.



THE EQUATION OF MOTION FOR TRANSVERSE VIBRATIONS IS EASILY FOUND. LET  $\rho =$  MASS/AREA ;  $s(x,y,t) =$  DISPLACEMENT

$$\text{THEN } \rho dx dy \ddot{s} = F_{\text{TRANSVERSE}}$$

THE SURFACE TENSION PULLS ON EACH OF THE 4 SIDES OF THE AREA ELEMENT  $dx dy$ .



WE WANT THE VERTICAL COMPONENT.

$$F_z = \underbrace{T dy}_{\text{FORCE ACROSS y EDGE}} \left( \underbrace{\frac{\partial S}{\partial x}(x+dx) - \frac{\partial S}{\partial x}(x)}_{\text{VERTICAL COMPONENTS}} \right) + T dx \left( \frac{\partial S}{\partial y}(y+dy) - \frac{\partial S}{\partial y}(y) \right)$$

$$= T dx dy \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right)$$

$$\text{so } \boxed{\rho \ddot{s} = T \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right) \equiv T (S_{xx} + S_{yy})}$$

THIS FOLLOWS ALSO FROM LAGRANGE'S METHOD:

$$I = \frac{1}{2} \iint \rho dx dy \dot{s}^2$$

$$V = T \left\{ \iint d\text{AREA} - A_0 \right\}$$

$$d\text{Area} \approx dx dy = \sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2} \sqrt{1 + \left(\frac{\partial S}{\partial y}\right)^2} dx dy \sim \left(1 + \frac{1}{2} S_x^2 + \frac{1}{2} S_y^2\right) dx dy$$

$$\text{so } V = \frac{T}{2} \iint (S_x^2 + S_y^2) dx dy$$

$$L = \frac{1}{2} \iint (\rho \dot{s}^2 - T S_x^2 - T S_y^2) dx dy$$

HAMILTON'S PRINCIPLE THEN QUICKLY GIVES

$$- \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{s}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial S_x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial S_y} = 0$$

$$\text{or } \underline{\rho \ddot{s} - T (S_{xx} + S_{yy}) = 0}$$

FOR A MEMBRANE TO SUPPORT A TENSION, WE SUPPOSE

$S=0$  ON THE BOUNDARY AS THE ONLY PHYSICALLY INTERESTING BOUNDARY CONDITION.

EXAMPLE RECTANGULAR MEMBRANE



WE CAN USE THE METHOD OF SEPARATION OF VARIABLES:

$$S(x, y, t) = f(x) g(y) h(t)$$

THE SOLUTION IS READILY SEEN TO BE

$$S_{m,n} = A_{m,n} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \omega_{m,n} t \quad (\text{or } \sin \omega t)$$

$$m, n = 1, 2, 3, \dots$$

BY PLUGGING INTO THE WAVE EQUATION, WE FIND

$$\omega_{m,n}^2 = \frac{T}{\rho} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right]$$

WE ILLUSTRATE THE MODES BY INDICATING THE LINES WHERE  $S = 0$  ALWAYS



SQUARE MEMBRANE

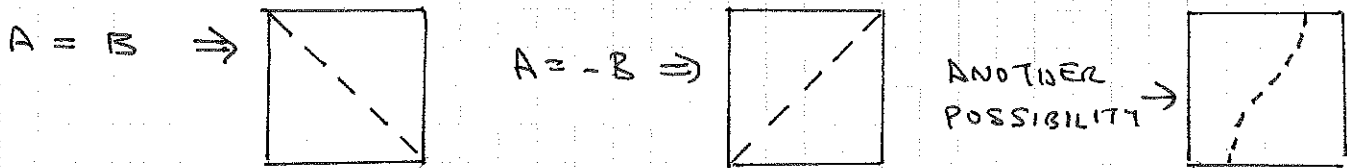
IF  $a = b$ , THEN  $\omega_{m,n} = \omega_{n,m}$

AND WE SAY THAT MODES  $(m,n)$  AND  $(n,m)$  ARE DEGENERATE.

THIS MEANS THAT  $A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} + B \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a}$

IS ALSO A MODE OF FREQUENCY  $\omega_{m,n}$  FOR ANY  $A \neq B$ .

EXAMPLE:  $m = 1, n = 2$



EXAMPLE  $m = 1, n = 3$

