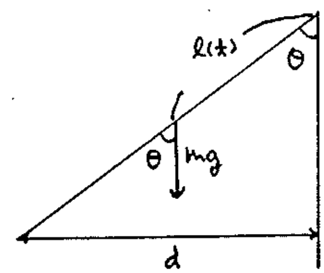
Ph 205 SET III






1. From the right figure, we find the object should move the distance

$$L = d/\sin\theta$$

The motion is determined by

$$m\frac{d^2}{dt^2}l(t) = mg\cos\theta \Rightarrow l(t) = \frac{1}{2}gt^2\cos\theta$$

where we assume the initial speed is 0. Thus the time of decent is given by

$$\frac{d}{\sin\theta} = \frac{1}{2}g\cos\theta\, t_d^2 \Rightarrow t_d = \sqrt{\frac{2d}{g\sin\theta\cos\theta}}$$

To minimize $t_d$, we require $\frac{d}{d\theta}t_d = 0$ to get $\sin^2\theta = \cos^2\theta$, which tells us that the optimal angle is $45^\circ$.

2. The expression for Surface area is given by

$$A = \int 2\pi x\, dl \quad \text{where } dl^2 = dx^2 + dy^2$$

which can directly be seen from the figure. Thus, our problem is to find the function y=y(x) that

$$A = \int 2\pi x\, dl = \int 2\pi x\sqrt{dx^2+dy^2} = \int_{x_1}^{x_2} 2\pi x\sqrt{1+y'^2}\,dx = \int_{x_1}^{x_2} L\,dx$$

is minimized. ( $y' = \frac{dy}{dx}$ ) From Euler-Lagrange equation,

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}L\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow \frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^2}}\right) = 0$$

The direct integration gives

$$\frac{xy'}{\sqrt{1+y'^2}} = A$$

which can be rearranged into

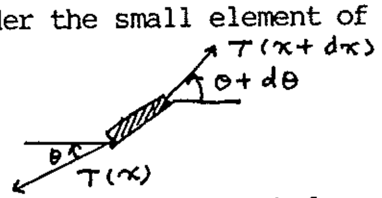
$$y' = A/\sqrt{x^2-A^2}$$

Thus,

$$y = \int \frac{A\,dx}{\sqrt{x^2-A^2}} = A\int d\theta = A\theta + B \Rightarrow \theta = \frac{y-B}{A} \Rightarrow x = A\cosh(y-B)/A$$

$$(x = A\cosh\theta)$$

3. We consider the small element of the rope.




From the horizontal force balance, we get

$$T(x+dx)\cos(\theta+d\theta) = T(x)\cos\theta$$

which yields, $\frac{d}{d\theta}T\cos\theta = 0 \Rightarrow T\cos\theta = T_o$ ;constant

From the vertial force balance we get

$$T(x+dx)\sin(\theta+d\theta) - T\sin\theta$$

$$= T_o\left(\frac{\sin(\theta+d\theta)}{\cos(\theta+d\theta)} - \frac{\sin\theta}{\cos\theta}\right)$$

$$= T_o\frac{d}{d\theta}\tan\theta\, d\theta = T_o\frac{d}{dx}\left(\frac{d}{dx}y\right)dx = T_o y''dx$$

$$= \rho g\, dl$$

where we used $\tan\theta = dy/dx$ and $\rho$ denotes line mass density of the rope. In this case

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx$$

Thus, we have

$$y''/\ 1+y' = \rho g/T_0 \quad ; \text{ constant}$$

(b) The only modification in this case is to replace $gdl$ with $gdx$. Thus, we have

$$y'' = \rho g/T_0 \ ; \text{ constant}$$

4. (a) We minimize the total energy

$$E = -\rho g \int_0^D \sqrt{1 + y'^2}\, y\, dx$$

under the constraint

$$\int_0^D \sqrt{1 + y'^2}\, dx = L.$$

Using the method of Lagrange multiplier, we consider

$$E^* = \rho g \left( -\int_0^D y \sqrt{1 + y'^2}\,dx + \lambda \int_0^D \sqrt{1 + y'^2}\, dx \right)$$

From Euler-Lagrange equation, we get, (cf. I'm trying to avoid $y'\frac{L}{y'} - L$ = constant, assuming you are familiar with that.)

$$\frac{d}{dx}\left( \frac{\partial}{\partial y'}\{(\lambda - y)\sqrt{1+y'^2}\}\right) - \frac{\partial}{\partial y}\{(\lambda - y)\sqrt{1+y'^2}\}$$

$$= \frac{d}{dx}\left(\frac{\lambda - y}{\sqrt{1+y'^2}}\right) + \sqrt{1+y'^2}$$

$$= (\lambda - y)\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) + \frac{1}{\sqrt{1+y'^2}} = 0 \qquad (1)$$

Let us set $y' = \sinh f$. Then

$$y = \int_0^x \sinh f\, dx + y(0) = \int_0^x \sinh f\, dx$$

Thus, (1) becomes,

$$\int_0^x \sinh f\, dx - \lambda = \frac{dx}{df}\cosh f \qquad (2)$$

We differentiate (2) with respect to x and get

$$\frac{d^2}{dx^2} f = 0$$

Thus, $f = (x + b)/c$. Consequently

$$y = \int_0^x \sinh (x+b)/c\, dx = c\,(\cosh (x+b)/c - \cosh b/c)$$

From the condition $y(D)=0$, we get $b = -D/2$. From the length constraint,

$$\int_0^D \sqrt{1+y'^2}\, dx = \int_0^D \sqrt{1+\sinh^2 f}\, dx = \int_0^D \cosh\frac{x - D/2}{c}\, dx = 2c\sinh\frac{D}{2c} = L$$

(comment. To students whom I recommended to see my calculations;
Some students interpreted $\lambda$ appeared above as an arbitrary reference point measuring gravitational potential. As is well known, the reference point of gravitational potentia can not appear in the actual path, I mean, in the actual physically directly measurable quantity. The change in the reference point should add some constant (only constant!) to Lagrangian which does not have any physical consequences. See also comment of problem 1c

e.g. mass, speed, position....

(b) In this case, we can write

$$E = -\rho g \int_0^D y\sqrt{1 + y'^2}\,dx - \quad g\, l^2/2$$

with constraint

$$L = \int_0^D \sqrt{1 + y'^2}\,dx + l \quad .$$

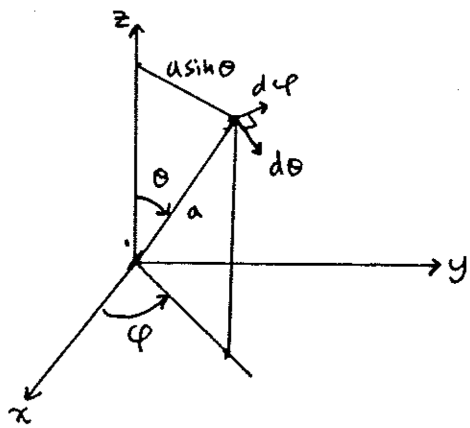
Thus, we minimize

$$E^* = \rho g \left( -\int_0^D \sqrt{1 + y'^2}\, y\, dx + \lambda \int_0^D \sqrt{1 + y'^2}\,dx - l^2/2 + \lambda\, l \right)$$

Note that l is an independent variable here. Hence,

$$\frac{d}{d\,l}\, E^* = 0 \qquad \text{implies} \qquad \lambda = l$$

The variational calculation along $0 < x < D$ is identical to the above calculation except for the length normalization,




$$2c \sinh D/2c = L - l$$

By setting x=0 in (2), $\lambda$ is determined to yield

$$\lambda = l = -c \cosh D/2c$$

(cf. Note that $c < 0$ in this case)

5. (a) In spherical polar coordinates,

$$ds^2 = a^2 \left( \sin^2\theta\, d\varphi^2 + d\theta^2 \right)$$

Thus the length is defined by

$$L = a \int_{\varphi_1}^{\varphi_2} \sqrt{\sin^2\theta + \theta'^2}\, d\varphi$$

where $\theta' = d\theta/d\varphi$ . Since $\frac{\partial}{\partial\varphi}\sqrt{\sin^2\theta + \theta'^2} = 0$, we know that

$$\theta' \frac{\partial}{\partial\theta'} L - L = \frac{\theta'^2}{\sqrt{\sin^2\theta + \theta'^2}} - \sqrt{\sin^2\theta + \theta'^2} = \frac{-\sin^2\theta}{\sqrt{\theta'^2 + \sin^2\theta}} = -C \text{ ; constant}$$

Thus,

$$d\varphi = \frac{d\theta}{\sin\theta\sqrt{\sin^2\theta - C^2}}$$

Using integral table,

$$\varphi - \varphi_0 = \cos^{-1}\left(\frac{c\cot\theta}{\sqrt{1-c^2}}\right)$$

By taking cosine of both sides we get,

$$\frac{c}{\sqrt{1-c^2}}\cos\theta = \sin\theta\cos(\varphi - \varphi_0) = \sin\theta\cos\varphi\cos\varphi_0 + \sin\theta\sin\varphi\sin\varphi_0 \qquad (1)$$

From the above figure,

$$x = a\sin\theta\cos\varphi \; , \; y = a\sin\theta\sin\varphi \, , \quad z = a\cos\theta$$

Hence (1) becomes

$$\frac{c}{\sqrt{1-c^2}}\, z = x\cos\varphi_0 + y\sin\varphi_0$$

,i.e., the equation of the arbitrary plane passing through the center of the sphere.

(b) We would like to minimize

$$I = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

under the constraint

$$g = x^2 + y^2 + z^2 - a^2 = 0$$

Using the method of Lagrange multiplier, the extremum condition is

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}} f\right) - \lambda \frac{\partial}{\partial x} g = 0 \quad \Rightarrow \quad \lambda = \frac{d}{dt}\left(\frac{\dot{x}}{f}\right) \Big/ \frac{\partial g}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}} f\right) - \lambda \frac{\partial}{\partial y} g = 0 \quad \Rightarrow \quad \lambda = \frac{d}{dt}\left(\frac{\dot{y}}{f}\right) \Big/ \frac{\partial g}{\partial y}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{z}} f\right) - \lambda \frac{\partial}{\partial z} g = 0 \quad \Rightarrow \quad \lambda = \frac{d}{dt}\left(\frac{\dot{z}}{f}\right) \Big/ \frac{\partial g}{\partial z}$$

where we used $\frac{\partial}{\partial \dot{x}} f = \dot{x} / \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = x/f$ , ..., etc. Above relations can be written as

$$\frac{d}{dt}\left(\frac{\dot{x}}{f}\right) \Big/ \frac{\partial g}{\partial x} = \frac{d}{dt}\left(\frac{\dot{y}}{f}\right) \Big/ \frac{\partial g}{\partial y} = \frac{d}{dt}\left(\frac{\dot{z}}{f}\right) \Big/ \frac{\partial g}{\partial z} = \lambda$$

In case of sphere,

$$\frac{1}{x} \frac{d}{dt}\left(\frac{\dot{x}}{f}\right) = \frac{1}{y} \frac{d}{dt}\left(\frac{\dot{y}}{f}\right) = \frac{1}{z} \frac{d}{dt}\left(\frac{\dot{z}}{f}\right) = k = 2\lambda.$$

Thus

$$\ddot{x} = k\, x\, f + \dot{x}\, \dot{f} / f \ , \ \ddot{y} = k\, y\, f + \dot{y}\, \dot{f} / f \ , \ \ddot{z} = k\, z\, f + \dot{z}\, \dot{f} / f \ .$$

Motivated by the observation of the above equation that it looks like central force case, we define following variables. (In fact, they look like angular momenta.)

$$L_z = x\dot{y} - y\dot{x} \ , \ L_y = z\dot{x} - x\dot{z} \ , \ L_z = y\dot{z} - z\dot{y}$$

By differentiating each L's, we get

$$\dot{L}_z = L_z\, \dot{f} / f \ , \ \dot{L}_y = L_y\, \dot{f} / f \ , \ \dot{L}_x = L_x\, \dot{f} / f$$

We directly integrate above equations to get

$$\ln L_z = \ln f + c1 \quad \Rightarrow \quad L_z = \text{constant} \cdot f \equiv X$$

Likewise,

$$L_y = AX \ , \ L_x = BX \qquad (A, B ; \text{constants})$$

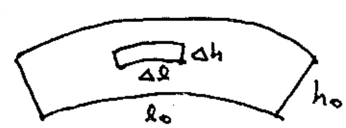
Also motivated by the antisymmetry of each L's, we calculate

$$\begin{aligned} x L_x + y L_y + z L_z &= x(y\dot{z} - z\dot{y}) + y(z\dot{x} - x\dot{z}) + z(x\dot{y} - y\dot{x}) \\ &= 0 \\ &= zX + Ay\, X + Bx\, X \end{aligned}$$

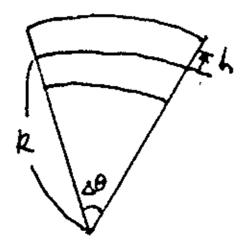
Thus, we finally get,

$$A\, y + B\, x + z = 0$$

6. If we put two identical springs together in series, the spring constant will be halved. In case of parallel connection, it will be doubled. Thus, the local spring constant of the infinitesmal piece shown in the figure is




$$k_{\text{local}} = k \frac{\Delta h}{h_0} \frac{\ell_0}{\Delta \ell}$$




The local variation of the length is

$$(R+h)\Delta\theta - R\Delta\theta = h\Delta\theta$$

Thus the local energy is given by

$$\Delta E = 1/2\ k_{local}\ (h\Delta\theta)^2 = 1/2\ k \frac{l_0}{h_0}\left(\frac{\Delta\theta}{\Delta l}\right)^2 h^2 \Delta l\, \Delta h$$

Now, we sum the whole pieces to get

$$\text{P.E.} = \int \Delta E = \int_0^{l_0} dl \int_{-h_0/2}^{h_0/2} \frac{k}{2}\frac{l_0}{h_0}\left(\frac{d\theta}{dl}\right)^2 h^2\, dh$$

$$= \frac{1}{24} k l_0 h_0^2 \int_0^{l_0}\left(\frac{d\theta}{dl}\right)^2 dl$$

We would like to minimize P.E. under the constraint

$$x = \int_0^{l_0} \cos\theta\, dl$$

We use the method of Lagrange multiplier. Thus,

$$\frac{d}{dl}\left(\frac{\partial}{\partial\theta'}\left(\frac{1}{24}k l_0 h_0^2 \theta'^2 + \lambda\cos\theta\right)\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{24}k l_0 h_0^2\theta'^2 + \lambda\cos\theta\right) = 0 \qquad \left(\theta' = \frac{d\theta}{dl}\right)$$

$$\Rightarrow \frac{d^2}{dl^2}\theta + \omega^2 \sin\theta = 0 \qquad (1)$$

where $\omega^2 = 12\lambda / k l_0 h_0^2$ . If $\theta$ is small, we can approximate above equation as,

$$\frac{d^2}{dl^2}\theta + \omega^2\theta \approx 0$$

We should additionally require that $\frac{d\theta}{dl} = 0$ at $l=0$ though it's not explicit in problem. We will assume it and the condition along with the $\theta = \theta_1$ at $l=0$ gives the simple solution,

$$\theta \approx \theta_1 \cos\omega l$$

The general form of $\theta(l)$ can be calculated as follows. By integrating (1),

$$\frac{1}{2}\left(\frac{d\theta}{dl}\right)^2 = \omega^2\cos\theta - c \quad \Rightarrow \quad d\theta / \sqrt{\cos\theta - \cos\theta_M} = \sqrt{2}\,\omega\, dl$$

where we defined $\cos\theta_M = c/\omega^2$. After the variable change $\sin(\theta/2) = \sin(\theta_M/2)\sin\varphi$ , we get (after lengthy calculations)

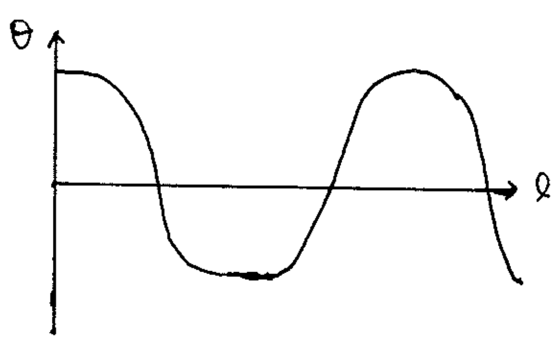
$$l = \frac{4}{\omega}\int_{\theta_1}^{\sin^{-1}(\sin\frac{\theta}{2}/\sin\frac{\theta_M}{2})} \left(1 - \sin^2\frac{\theta_M}{2}\sin^2\varphi\right)^{-1/2} d\varphi$$

$$= \frac{4}{\omega}\left(F\left(\sin^{-1}\left(\sin\frac{\theta}{2}/\sin\frac{\theta_M}{2}\right) \backslash \theta_M\right) - F(\theta_1 \backslash \theta_M)\right)$$

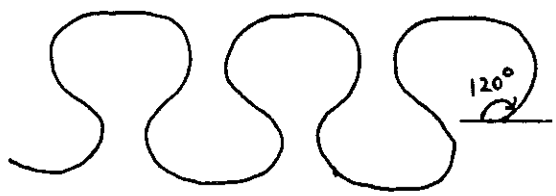
where I used

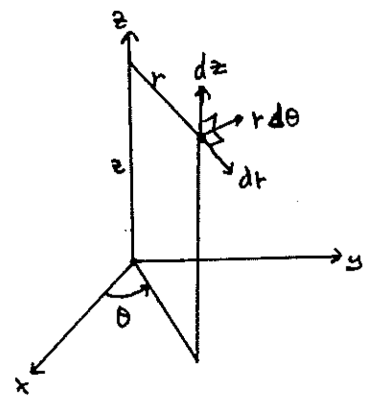
$$F(\varphi\backslash\alpha) = \int_0^{\varphi} (1 - \sin^2\alpha\,\sin^2\theta)^{-1/2}\, d\theta$$

incomplete elliptic integral. (See p.322 of Arfken, "Mathematical Methods for Physicists" for detailed discussions and other sources of information.) Although the analytic formula looks quite complicated, the qualitative feature can easily be understood. (1) is a harmonic oscillator with the varying spring constant. If $\theta$ is large, the spring constant is smaller. Thus we infer that there will be periodic function which varies slowly than usual trigonometric functions for large amplitude. Thus, the graph looks like



Thus for $\theta_1 = 120^\circ$, the curve looks like




7. (a) In cylindrical coordinate systems, the length element can be written as

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Thus, the kinetic energy can be written as,

$$T = 1/2\ m\ (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

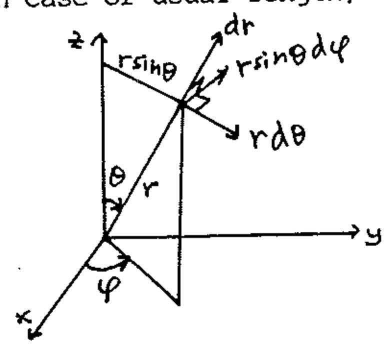
We directly find that $f_r = 1$, $f_\theta = r$, and $f_z = 1$. By the straightforward computations,

$$a_r = \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r}\right)/mf_r = (m\ddot{r} - 2mr\dot{\theta}^2)/m = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = m(2r\dot{r} + r^2\ddot{\theta})/mr = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

$$a = m\ddot{z}/m = \ddot{z}$$

(cf. From the above figure, you can directly read the length element since cylindrical coordinates are orthogonal coordinates. If a coordinate measures angle, the corresponding f is just the arm length. Of course, in case of usual length, we immediately find that f=1.)




(b) In spherical coordinates, the length element can be written as,

$$ds^2 = dr^2 + r^2\sin^2\theta\, d\varphi^2 + r^2 d\theta^2$$

Thus, the kinetic energy is

$$T = 1/2\ m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2)$$

which indicates $f_r = 1$, $f_\theta = r$, and $f_\varphi = r\sin\theta$. By the same calculations,

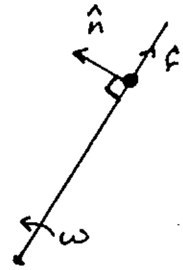
$$a_r = m(\ddot{r} - r\sin^2\theta\,\dot{\varphi}^2 - r\dot{\theta}^2)/m = \ddot{r} - r\sin\theta\,\dot{\varphi}^2 - r\dot{\theta}^2$$

$$a_\theta = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta} - r^2\sin\theta\cos\varphi\,\dot{\varphi}^2)/mr = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2$$

$$a_\varphi = m(2r\dot{r}\sin^2\theta\,\dot{\varphi} + 2r^2\sin\theta\cos\theta\,\dot{\theta}\dot{\varphi} + r^2\sin^2\theta\,\ddot{\varphi})/mr\sin\theta = 2\dot{r}\sin\theta\,\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi} + r\sin\theta\,\ddot{\varphi}$$

8. (a) Elementary Method

Since the bead can move freely along $\hat{r}$ direction, the constraint force in this case should act along $\hat{n}$ direction.




The total kinetic energy can be written as,

$$T = 1/2\ m\ r^2\omega^2 + 1/2\ m\ \dot{r}^2$$

Thus,

$$\frac{dT}{dt} = mr\omega^2\dot{r} + m\dot{r}\ddot{r} = 2mr\omega^2\dot{r}$$

The constraint force, force $\vec{F}$, should be responsible for this change in energy. Therefore,

$$\vec{F}\cdot\vec{v} = F\cdot r\omega = 2mr\omega^2\dot{r} \quad \Rightarrow F = 2m\dot{r}\omega$$

(b) Lagrange Multiplier Method

The constraint in this case is $\dot{\theta} = \omega$. Thus, $a_\theta = 1$ and $a_r = 0$.

The Lagrangian is,

$$L = 1/2\ m\ (\dot{r}^2 + r^2\dot{\theta}^2)$$

Thus, Euler-Lagrange equation gives,

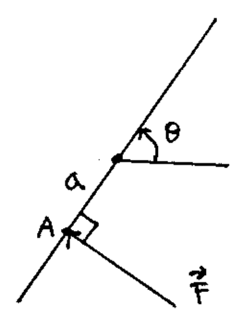
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial}{\partial r}L = \lambda a_r \quad \Rightarrow \quad \ddot{r} = \dot{\theta}^2 r \Rightarrow \ddot{r} = \omega^2 r$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda a_\theta \quad \Rightarrow \quad \lambda = mr\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 2mr\dot{r}\dot{\theta}$$

We can interpret $\lambda$ as a torque exerted by F. (You can readily verify this assertion using the definition of generalized force.)

$$F = \lambda / r = 2m\dot{r}\omega$$

9. Let us consider the speed at point A. It consists of two parts due to the translational motion of CM and the instantaneous rotation about CM induced by $\vec{F}$. Thus,




$$\vec{v} = \vec{v}_{cm} - a\dot{\theta}\,\hat{F}$$

Since F can do no work,

$$\vec{F}\cdot\vec{v} = \vec{F}\cdot\vec{V}_{cm} - a\dot{\theta}F$$

$$= F_x(\vec{v}_{cm}\cdot\hat{x}) + F_y(\vec{v}_{cm}\cdot\hat{y}) - a\dot{\theta}F$$

$$= F\left(-\sin\theta\frac{dx}{dt} + \cos\theta\frac{dy}{dt} - a\dot{\theta}\right) = 0$$

Hence, we have the constraint

$$a\ d\theta = -\sin\theta\ dx + \cos\theta\ dy \qquad (1)$$

The Lagrangian in this case is,

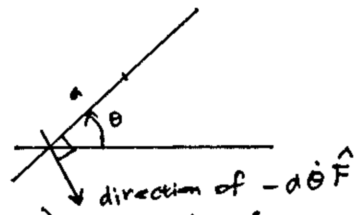
L = T = translational + rotaitional

$$= 1/2\ m\ (\dot{x}^2 + \dot{y}^2) + 1/2\ I\ \dot{\theta}^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}mb^2\dot{\theta}^2$$

Consequently, the Lagrange equation becomes, under constraint (1),

$$\ddot{x} = \lambda\sin\theta, \quad \ddot{y} = -\lambda\cos\theta, \quad \ddot{\theta} = a/b^2\ \lambda \qquad (2)$$

Referring to the right figure, the velocity of the point A reads




$$x\text{ - component ; } v\cos\theta = \dot{x} + a\sin\theta\,\dot{\theta} \qquad y\text{ - component ; } v\sin\theta = \dot{y} - a\cos\theta\,\dot{\theta} \tag{3}$$

Combining (2) and (3), we get

$$\ddot{x} = \dot{v}\cos\theta - v\sin\theta\,\dot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta = \sin\theta\,\lambda \tag{4}$$

$$\ddot{y} = \dot{v}\sin\theta + v\cos\theta\,\dot{\theta} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = -\cos\theta\,\lambda \tag{5}$$

Now, (4) $\cos\theta$ + (5) $\sin\theta$ and $-$ (4) $\sin\theta$ + (5) $\cos\theta$ give,

$$\dot{v} = a\dot{\theta}^2 \quad , \qquad ak^2\ddot{\theta} + v\dot{\theta} = 0$$

where $k = 1 + b^2/a^2$

Combining these two equations yields,

$$-k^2\frac{d}{dt}\left(\frac{\ddot{\theta}}{\dot{\theta}}\right) = \dot{\theta}^2 \Rightarrow -k^2\frac{\ddot{\theta}}{\dot{\theta}}\frac{d}{dt}\left(\frac{\ddot{\theta}}{\dot{\theta}}\right) = \dot{\theta}\ddot{\theta} \Rightarrow -k^2\left(\frac{\ddot{\theta}}{\dot{\theta}}\right)^2 = \dot{\theta}^2 + c^2 \Rightarrow k\frac{d\dot{\theta}}{\dot{\theta}\sqrt{c^2-\dot{\theta}^2}} = dt \tag{6}$$

We integrate above equation as follows,

$$\underset{(z=1/\dot{\theta})}{\int\frac{k\,d\dot{\theta}}{\dot{\theta}\sqrt{c^2-\dot{\theta}^2}}} = -k\underset{(cz=\cosh\alpha)}{\int\frac{dz}{\sqrt{c^2z^2-1}}} = -\frac{k}{c}\int d\alpha = -\frac{k}{c}\alpha = t - t_0 \Rightarrow z = \frac{1}{c}\cosh\frac{c}{k}(t-t_0)$$

Thus,

$$\dot{\theta} = \frac{c}{\cosh c(t-t_0)/k} = \frac{c}{\cosh\frac{ct}{k}}$$

where we set $t_0 = 0$. (6) can also be integrated as follows.

$$\frac{k\,d\dot{\theta}}{\sqrt{c^2-\dot{\theta}^2}} = \dot{\theta}\,dt \Rightarrow k\int\frac{d\dot{\theta}}{\sqrt{c^2-\dot{\theta}^2}} = \int d\theta \underset{(\dot{\theta}=c\cos\alpha)}{\Rightarrow} -k\int d\alpha = \int d\theta \Rightarrow \dot{\theta} = C\cos\frac{1}{k}(\theta-\theta_0)$$

If we require $\theta = 0$ at $t = 0$, we get

$$\theta_0 = 0$$

$$\therefore\ \underline{\dot{\theta} = \frac{c}{\cosh\frac{ct}{k}} = C\cos\frac{\theta}{k}}. \tag{7}$$

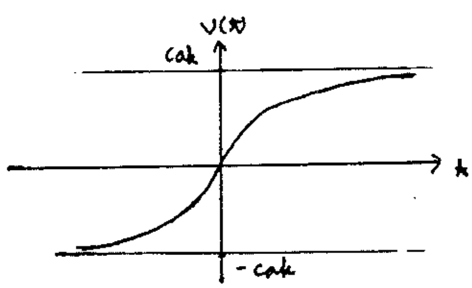
Now, v can be written as,

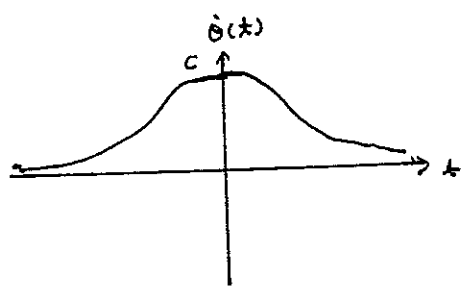
$$v = -ak^2\ddot{\theta}/\dot{\theta} = cak\sin\theta/k \quad \left(\text{From (7), } \ddot{\theta} = -c/k\,\sin\theta/k\ \dot{\theta}\right) \tag{8}$$

$$= cak\sqrt{1-\cos^2\theta/k} = cak\sqrt{\cosh^2 ct/k - 1}\,/\cosh ct/k)$$

$$= cak\tanh ct/k$$

Thus, the graph of v and $\dot{\theta}$ is given by,



From (7) and (8), we obtain tan $\theta$ /k = sinh ct/k which shows that if t is large, $\theta = \frac{\pi}{2}k$ and that if t is negatively small, $\theta = -\frac{\pi}{2}k$.

Now, let us consider the behavior near t = 0. Notice that x and y appearing in the problem set denote the coordinates of the point A. Thus,

$$\frac{dx}{d\theta} = \frac{dx}{dt}\frac{dt}{d\theta} = v\cos\theta \cdot \frac{1}{\dot\theta} = ak\tan\theta/k\cos\theta$$

$$\frac{dy}{d\theta} = \frac{dy}{dt}\frac{dt}{d\theta} = v\sin\theta \cdot \frac{1}{\dot\theta} = ak\tan\theta/k\sin\theta$$

where we used v=caksin $\theta$ /k and $\dot\theta$ =c cos$\theta$ /k.We expand both of above quantities using Taylor expansion. The leading order gives, ( $\theta = 0$ when $t = 0$)

$$\frac{dx}{d\theta} \approx a\theta, \quad \frac{d}{d\theta}y \approx a\theta^2$$

Thus, we immediately find that (at $t=0$)

$$\frac{dx}{d\theta} = \frac{d}{d\theta}y = \frac{d^2}{d\theta^2}y = 0 \quad \text{and} \quad \frac{d^2}{d\theta^2}x = a \neq 0, \quad \frac{d^3}{d\theta^3}y = 2a \neq 0.$$

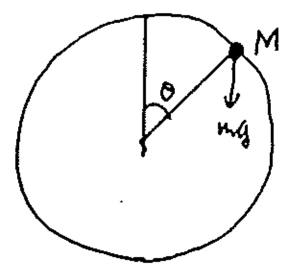
By the straightforward calculations, we also find that

$$\begin{aligned}\text{Kinetic Energy} &= \tfrac{1}{2}m(\dot x^2 + \dot y^2) + \tfrac{1}{2}mb^2\dot\theta^2 \\ &= \tfrac{1}{2}mv^2 + \tfrac{m}{2}k^2a^2\dot\theta^2 \\ &= \tfrac{1}{2}mc^2a^2k^2\end{aligned}$$

and

$$F = \lambda = -m(v\dot\theta + a\ddot\theta) = -\frac{m}{2}\frac{c^2b^2}{ak}\sin^2\frac{\theta}{k}.$$

10. (a) The Lagrangian can be written as,




$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - mgr\cos\theta$$

with the constraint

r = constant

Thus, Euler-Lagrange equation can be written as

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot r}L\right) - \frac{\partial}{\partial r}L = m\ddot r - mr\dot\theta^2 + mg\cos\theta = \lambda \quad (1)$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot\theta}L\right) - \frac{\partial}{\partial\theta}L = m\frac{d}{dt}(r^2\dot\theta) - mgr\sin\theta = 0 \quad (2)$$

From (2), $\ddot\theta = \frac{g}{r}\sin\theta$. We integrate this to get,

$$\frac{1}{2}\dot\theta^2 = \frac{g}{r}(1-\cos\theta)$$

where we used the fact $\dot\theta$ =0 at t=0, $\theta$ =0. Thus, from (1),

$$\lambda = mg\cos\theta - mr\dot\theta^2 = mg(3\cos\theta - 2)$$

We can interpret this $\lambda$ as a normal force. Hence, $\lambda$ =0 gives,

$$\cos\theta = \frac{2}{3}$$

Comment. To those whom I recommended to see my calculations,

Consider the following argument.

" L = T - V. Assume that T and V don't have the explicit time dependence. Furthermore, if T consists only of quadratic terms in generalized velocity, the total energy T + V = E is conserved. That is L = T + V - 2V = E - 2V where E is constant, say 1 Joule, for example. Then, Lagrange's equation gives

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}L\right) - \frac{\partial}{\partial x}L = 2\frac{\partial}{\partial x}V = 0 \ ??$$

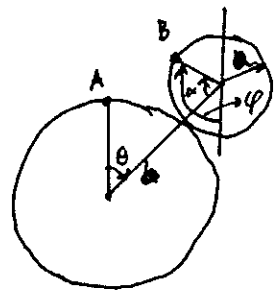
Your mistake is basically identical to this (apparently) wrong arguments!

(b) The constraint in this case is

$$b\dot{\theta} = a(\dot{\varphi} - \dot{\theta}) \qquad (1)$$

and

$$r = a+b \ ; \text{constant.} \qquad (2)$$




because $b\dot{\theta} = a\dot{\alpha}$ (rolling without slipping) and $\varphi = \alpha + \theta$. Now Lagrangian is given by

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}I\dot{\varphi}^2 - mgr\cos\theta$$

Since we are interested only in normal force, we consider only one Lagrange multiplier concerned with (2). (Caution! This is possible only when two constraints are completely decoupled as in our case. Think about the detailed reason why this ad hoc omission of one Lagrange multiplier works.) And from constraint (1), we rewrite our Lagrangian as,

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\frac{(a+b)^2}{a^2}\dot{\theta}^2 - mgr\cos\theta$$

(Caution! It is very important to note that I write in one place a+b and in another place r though they seem identical in view of (2). They are NOT identical.)

Thus, Euler-Lagrange equation gives,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda \qquad (3)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m\frac{d}{dt}\left(r^2\dot{\theta} + \frac{2}{5}(a+b)^2\dot{\theta}\right) - mgr\sin\theta = 0$$

where we used the fact that the moment of inertia of sphere is

$I = 2/5\ ma^2$

Using (2), (3) becomes,

$$\frac{7}{5}\ddot{\theta} = \frac{g}{r}\sin\theta \quad \Rightarrow \quad \dot{\theta}^2 = \frac{10}{7}\frac{g}{r}(1-\cos\theta) \qquad (\dot{\theta} = 0 \text{ when } t=0)$$

$$\lambda = mg\left(\cos\theta - \frac{r}{g}\dot{\theta}^2\right) = mg\left(\frac{17}{7}\cos\theta - \frac{10}{7}\right)$$

Now, $\lambda=0$ gives,

$$\cos\theta = \frac{10}{17}.$$

(cf. For more detailed explanations, see p.374 ~ 378. of "Mechanics", Symon, 3rd edition.)