# Potentials for a Rectangular Electromagnetic Cavity

Kirk T. McDonald
*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544*

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## 1 Problem

Deduce scalar and vector potentials relevant to electromagnetic modes in a rectangular cavity of dimensions $d_x \ge d_y \ge d_z$, assuming the walls to be perfect conductors, and the interior of the cavity to be vacuum.

## 2 Solution

*For the case of a cylindrical cavity, see [1].*

### 2.1 E and B Fields of the Cavity Modes

The cavity has extent $0 < x < d_x$, $0 < y < d_y$, and $0 < z < d_z$. The electric field must be everywhere perpendicular to the (perfectly conducting) walls, such that for time dependence $e^{-i\omega t}$ there exists a set of modes with non-negative integer indices $\{l, m, n\}$ of the form,

$$E_x = E_0\, e_x \cos k_x x \sin k_y y \sin k_z z\, e^{-i\omega t}, \tag{1}$$

$$E_y = E_0\, e_y \sin k_x x \cos k_y y \sin k_z z\, e^{-i\omega t}, \tag{2}$$

$$E_z = E_0\, e_z \sin k_x x \sin k_y y \cos k_z z\, e^{-i\omega t}, \tag{3}$$

where $\hat{\mathbf{e}} = (e_x, e_y, e_z)$ is a unit vector, the wave vector $\mathbf{k}$ is given by,

$$\mathbf{k} = (k_x, k_y, k_z) = \pi \left( \frac{l}{d_x}, \frac{m}{d_y}, \frac{n}{d_z} \right), \tag{4}$$

and at most only one of indices $l$, $m$, or $n$ is zero. These fields obey the free-space wave equation,

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{\omega^2}{c^2} \mathbf{E}, \tag{5}$$

where $c$ is the speed of light in vacuum, which implies that,

$$\omega = kc = \pi c \sqrt{\frac{l^2}{d_x^2} + \frac{m^2}{d_y^2} + \frac{n^2}{d_z^2}}. \tag{6}$$

The first (free-space) Maxwell equation, $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ implies that $\hat{\mathbf{e}} \cdot \mathbf{k} = 0$, so that there are two orthogonal "polarizations" $\hat{\mathbf{e}}$ for each set of indices $\{l, m, n\}$.[1]

The magnetic field is related to the electric field by Faraday's law (in Gaussian units),

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial ct} = ik\mathbf{B}, \tag{8}$$

such that

$$\begin{aligned} B_x &= iE_0\, b_x \sin k_x x \cos k_y y \cos k_z z\, e^{-i\omega t}, & (9)\\ B_y &= iE_0\, b_y \cos k_x x \sin k_y y \cos k_z z\, e^{-i\omega t}, & (10)\\ B_z &= iE_0\, b_z \cos k_x x \cos k_y y \sin k_z z\, e^{-i\omega t}, & (11) \end{aligned}$$

where $\hat{\mathbf{b}}$ is the unit vector,

$$\hat{\mathbf{b}} = \hat{\mathbf{e}} \times \hat{\mathbf{k}} = \frac{1}{k}(e_y k_z - e_z k_y, e_z k_x - e_x k_z, e_x k_y - e_y k_x). \tag{12}$$

The magnetic field is everywhere tangential to the walls of the cavity (which motivated the use of the cosine functions in the electric field (1)-(3)). Thus, $\hat{\mathbf{b}} \cdot \mathbf{k} \propto \det(\hat{\mathbf{e}}, \mathbf{k}, \mathbf{k}) = 0$, consistent with the third Maxwell equation, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. Also, $\hat{\mathbf{e}} \cdot \hat{\mathbf{b}} \propto \det(\hat{\mathbf{e}}, \hat{\mathbf{e}}, \mathbf{k}) = 0$, such that for each mode the vectors $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{k}$ form a mutually orthogonal triad, with,

$$\hat{\mathbf{e}} = \hat{\mathbf{b}} \times \hat{\mathbf{k}} = \frac{1}{k}(b_y k_z - b_z k_y, b_z k_x - b_x k_z, b_x k_y - b_y k_x). \tag{13}$$

## 2.2 Potentials

The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be related to scalar and vector potentials $V$ and $\mathbf{A}$ according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla} V + ik\mathbf{A}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{14}$$

In sec. 2.1 we deduced the electromagnetic fields inside the cavity, but did not comment on their values outside it. In the following, we suppose that $\mathbf{E}$ and $\mathbf{B}$ are zero outside the cavity.

[1] The electric field can be regarded as the superposition of eight plane waves,

$$\begin{aligned} \mathbf{E} \;=\; -\frac{E_0}{8}\Big[ &(e_x, e_y, e_z)\, e^{i(k_x x + k_y y + k_z z - \omega t)} + (-e_x, e_y, e_z)\, e^{i(-k_x x + k_y y + k_z z - \omega t)} \\ &-(e_x, -e_y, e_z)\, e^{i(k_x x - k_y y + k_z z - \omega t)} + (-e_x, -e_y, e_z)\, e^{i(-k_x x - k_y y + k_z z - \omega t)} \\ &-(e_x, e_y, -e_z)\, e^{i(k_x x + k_y y - k_z z - \omega t)} - (-e_x, e_y, -e_z)\, e^{i(-k_x x + k_y y - k_z z - \omega t)} \\ &+(e_x, -e_y, -e_z)\, e^{i(k_x x - k_y y - k_z z - \omega t)} + (-e_x, -e_y, -e_z)\, e^{i(-k_x x - k_y y - k_z z - \omega t)}\Big], \end{aligned} \tag{7}$$

with a similar relation holding for the magnetic field.

### 2.2.1 Hamiltonian Gauge

A simple option for the potentials is to adopt the so-called Hamiltonian gauge in which the scalar potential is everywhere zero (see, for example, sec. 8 of [2]),[2,3]

$$V^{(\mathrm{H})} = 0, \qquad \mathbf{A}^{(\mathrm{H})} = -\frac{i\mathbf{E}}{k}\,. \tag{15}$$

Then, $\boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{H})} = \mathbf{B}$ is confirmed by use of Faraday's law, eq. (8).

This vector potential is not continuous on the planar faces of the cavity. However, this is not a formal problem in that the computation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ next to the surface does not involve derivatives normal to that surface.[4]

### 2.2.2 Poincaré Gauge

In cases where the $\mathbf{E}$ and $\mathbf{B}$ fields are known, we can compute the potentials in the so-called Poincaré gauge (see sec. 9A of [2] and [6, 7]),[5]

$$V^{(\mathrm{P})}(\mathbf{x}, t) = -\mathbf{x} \cdot \int_0^{u_0=1} du\, \mathbf{E}(u\mathbf{x}, t), \qquad \mathbf{A}^{(\mathrm{P})}(\mathbf{x}, t) = -\mathbf{x} \times \mathbf{I}(\mathbf{B}), \tag{16}$$

where,

$$\mathbf{I}(\mathbf{F}) = \int_0^{u_0=1} u\, du\, \mathbf{F}(u\mathbf{x}, t) = u_0 \mathbf{G}(u_0\mathbf{x}, t) - \int_0^{u_0=1} du\, \mathbf{G}(u\mathbf{x}, t) \tag{17}$$

and,

$$\mathbf{F}(u\mathbf{x}, t) = \frac{d\mathbf{G}(u\mathbf{x}, t)}{du}\,. \tag{18}$$

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.[6]

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[2]This gauge appears to have been first used by Gibbs in 1896 [3].

[3]For a static electric field the Hamiltonian-gauge vector potential is $\mathbf{A}^{(\mathrm{H})} = -c(t - t_0)\mathbf{E}$, while for a static magnetic field the vector potential is the same as that in the Coulomb gauge (and also in the Lorenz gauge).

[4]In Hamiltonian dynamics of a particle with charge $q$ the normal ($z$) component of the canonical momentum $\mathbf{p} = \mathbf{p}_{\mathrm{mech}} + q\mathbf{A}/c$ takes a discontinuous step when a particle enters or exits the rf cavity through the planar faces. This undesirable feature can be mitigated by switching from coordinates $(x, y, z)$ with independent variable $t$ to coordinates $(x, y, t)$ with independent variable $z$, in which case the canonical momentum of the $t$-coordinate is $p_t = -E_{\mathrm{mech}} - qV$ (see, for example, sec. 1.6 of [5]), which is just $-E_{\mathrm{mech}}$ in the Hamiltonian gauge. Then, if the faces of the cavity traversed by particles are at constant $z$, all three canonical momenta $p_x$, $p_y$ and $p_t$ are continuous. *Only if the particles are muons would it be considered practical to used closed rf cavities for their acceleration.*

[5]The Poincaré gauge is also called the multipolar gauge [8].

[6]The potentials in the Poincaré gauge depend on the choice of origin. If the origin is inside the region of electromagnetic fields, then the Poincaré potentials are nonzero throughout all space. If the origin is to one side of the region of electromagnetic fields, then the Poincaré potentials are nonzero only inside that region, and in the region on the "other side" from the origin.

For points $\mathbf{x}$ outside the cavity, we restrict the calculation to the case that the vector $\mathbf{x}$ passes through the cavity wall at $z = d_z$, along which $\mathbf{E}(u\mathbf{x})$ and $\mathbf{B}(u\mathbf{x})$ are nonzero only for $u < u_0 = d_z/z$.

Using integrals 2.533.4-5 of [9] we have that,

$$\begin{aligned} -x\int_0^{u_0} du\, E_x &= -xE_0 e_x\, e^{-i\omega t}\int_0^{u_0} du\, \cos k_x ux \sin k_y uy \sin k_z uz \\ &= \frac{xE_0 e_x}{4}\, e^{-i\omega t}\left[\frac{\sin(k_x x + k_y y + k_z z)u_0}{k_x x + k_y y - k_z z} + \frac{\sin(-k_x x + k_y y + k_z z)u_0}{-k_x x + k_y y + k_z z}\right. \\ &\qquad \left. - \frac{\sin(k_x x - k_y y + k_z z)u_0}{k_x x - k_y y + k_z z} - \frac{\sin(k_x x + k_y y - k_z z)u_0}{k_x x + k_y y - k_z z}\right], \end{aligned} \tag{19}$$

where $u_0 = 1$ for $\mathbf{x}$ inside the cavity, and $u_0 = d_z/z$ for $\mathbf{x}$ outside the cavity such that vector $\mathbf{x}$ passes through the cavity face at $z = d_z$. Similar expressions hold for the terms $-yI_y(\mathbf{E})$ and $-zI_z(\mathbf{E})$ of the scalar potential $V^{(\mathrm{P})}$. At large $|\mathbf{x}|$ these terms fall of as $1/|\mathbf{x}|^2$, so the scalar potential, $V^{(\mathrm{P})} = -\mathbf{x}\cdot\mathbf{I}(\mathbf{E})$, has a dipole character.

For the vector potential we have that,

$$\begin{aligned} I_x(\mathbf{B}) &= iE_0 b_x\, e^{-i\omega t}\int_0^{u_0} u\, du\, \sin k_x ux \cos k_y uy \cos k_z uz \\ &= \frac{iE_0 b_x}{4}\, e^{-i\omega t}\left[-\frac{u_0\, \cos(k_x x + k_y y + k_z z)u_0}{k_x x + k_y y + k_z z} + \frac{u_0\, \cos(-k_x x + k_y y + k_z z)u_0}{-k_x x + k_y y + k_z z}\right. \\ &\qquad - \frac{u_0\, \cos(k_x x - k_y y + k_z z)u_0}{k_x x - k_y y + k_z z} - \frac{u_0\, \cos(k_x x + k_y y - k_z z)u_0}{k_x x + k_y y - k_z z} \\ &\qquad + \frac{\sin(k_x x + k_y y + k_z z)u_0}{(k_x x + k_y y + k_z z)^2} - \frac{\sin(-k_x x + k_y y + k_z z)u_0}{(-k_x x + k_y y + k_z z)^2} \\ &\qquad \left. + \frac{\sin(k_x x - k_y y + k_z z)u_0}{(k_x x - k_y y + k_z z)^2} + \frac{\sin(k_x x + k_y y - k_z z)u_0}{(k_x x + k_y y - k_z z)^2}\right], \end{aligned} \tag{20}$$

and similarly for $I_y(\mathbf{B})$ and $I_z(\mathbf{B})$. These integrals fall off at large $|\mathbf{x}|$ as $1/|\mathbf{x}|^2$, so the vector potential, $\mathbf{A}^{(\mathrm{P})} = -\mathbf{x}\times\mathbf{I}(\mathbf{B})$, falls off as $1/|\mathbf{x}|$.

### 2.2.3 Lorenz Gauge

In the Lorenz gauge the potentials are related by,

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{L})} = -\frac{1}{c}\frac{\partial V^{(\mathrm{L})}}{\partial t} = ikV^{(\mathrm{L})}, \tag{21}$$

where the latter form holds for time dependence $e^{-i\omega t}$. Then, the potentials obey the Helmholtz wave equations,

$$(\nabla^2 + k^2)\mathbf{A}^{(\mathrm{L})} = -\frac{4\pi}{c}\mathbf{J}, \qquad (\nabla^2 + k^2)V^{(\mathrm{L})} = -\frac{4\pi}{c}\rho, \tag{22}$$

whose formal solutions are the retarded potentials in the frequency domain,

$$V^{(\mathrm{L})}(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')\, e^{ikr}}{r}\, d\mathrm{Vol}', \qquad \mathbf{A}^{(\mathrm{L})}(\mathbf{x}) = \int \frac{\mathbf{J}(\mathbf{x}')\, e^{ikr}}{cr}\, d\mathrm{Vol}'. \tag{23}$$

The Lorenz gauge is a special case of a so-called velocity gauge,[7] *i.e.*, where the gauge condition is, in general,

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(v)} = -\frac{c}{v^2}\frac{\partial V^{(v)}}{\partial t} = i\frac{c^2}{v^2}kV^{(v)}, \tag{24}$$

while $v = c$ for the Lorenz gauge (and $v = \infty$ in the Coulomb gauge). The velocity-gauge potentials obey the differential equations,

$$\nabla^2 V^{(\mathrm{v})} - \frac{1}{v^2}\frac{\partial^2 V^{(\mathrm{v})}}{\partial t^2} = -4\pi\rho, \tag{25}$$

$$\nabla^2\mathbf{A}^{(\mathrm{v})} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}^{(\mathrm{v})}}{\partial t^2} = -\frac{4\pi}{c}\left[\mathbf{J} + \frac{1}{4\pi}\left(\frac{c^2}{v^2} - 1\right)\boldsymbol{\nabla}\frac{\partial V^{(\mathrm{v})}}{\partial t}\right]. \tag{26}$$

and a formal solution of the scalar potential in the velocity gauge is,

$$V^{(\mathrm{v})}(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}', t' = t - |\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}\, d\mathrm{Vol}', \tag{27}$$

so the scalar potential $V^{(\mathrm{v})}$ can be said to propagate with speed $v$.
Velocity-gauge potentials are not unique (for a given set of charges and currents), in that use of a *restricted* gauge-transformation function $\chi(\mathbf{x},t)$ which obeys $\nabla^2\chi - \partial^2\chi/\partial(vt)^2 = 0$, leads to new potentials,

$$V'^{(v)} = V^{(v)} - \frac{1}{c}\frac{\partial\chi}{\partial t}, \qquad \mathbf{A}'^{(v)} = \mathbf{A}^{(v)} + \boldsymbol{\nabla}\chi, \tag{28}$$

that also satisfy the condition (24).

For a cavity with perfectly conducting walls, the only charge and current densities reside on these walls, so the retarded potentials (23) have the form,

$$V^{(\mathrm{L,ret})}(\mathbf{x}) = \int \frac{\sigma(\mathbf{x}')\,e^{ikr}}{r}\, d\mathrm{Area}' = \int \frac{\mathbf{E}(\mathbf{x}')\cdot\hat{\mathbf{n}}'\,e^{ikr}}{4\pi r}\, d\mathrm{Area}', \tag{29}$$

$$\mathbf{A}^{(\mathrm{L,ret})}(\mathbf{x}) = \int \frac{\mathbf{K}(\mathbf{x}')\,e^{ikr}}{cr}\, d\mathrm{Area}' = \int \frac{\hat{\mathbf{n}}'\times\mathbf{B}(\mathbf{x}')\,e^{ikr}}{4\pi r}\, d\mathrm{Area}', \tag{30}$$

where $\sigma$ and $\mathbf{K}$ are the surface charge and current densities, and $\hat{\mathbf{n}}'$ is the inward unit vector normal to the bounding surface. These potentials are nonzero both inside and outside of the cavity.

However, it not easy to deduce the (retarded) Lorenz-gauge potentials for the present example from eq. (30). Instead, we follow a suggestion in Prob. 14.2 of [4], and note that the Hamiltonian-gauge potentials (6) satisfy the Lorenz-gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{L})} = -\partial V^{(\mathrm{L})}/\partial ct$ (and the vector potential satisfies the Coulomb-gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})} = 0$) inside the cavity, although the spatial derivatives are not defined on the planar cavity walls.[8] Are the Hamiltonian-gauge potentials truly also Lorenz-gauge potentials in the sense of satisfying the gauge condition (21) everywhere and not just inside the cavity?

Inside a rectangular cavity the wave equation (21) for the Lorenz-gauge vector potential is simply,

$$(\nabla^2 + k^2)\mathbf{A}^{(\mathrm{L})} = 0, \tag{31}$$

[8]Another possible motivation that leads to eq. (6) as the Lorenz- (or Coulomb-) gauge potentials for the cavity is the requirement that the tangential component of the vector potential vanish at the cavity walls.

which permits solutions where the components of the vector potential have the form,

$$A_j^{(\mathrm{L})} = \left\{ \begin{array}{c} \cos k_x x \\ \\ \sin k_x x \end{array} \right\} \left\{ \begin{array}{c} \cos k_y y \\ \\ \sin k_y y \end{array} \right\} \left\{ \begin{array}{c} \cos k_z z \\ \\ \sin k_z z \end{array} \right\}. \tag{32}$$

We can consider the special case that the vector potential (32) has only a $z$-component. Then,

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{L})} = \frac{\partial A_z^{(\mathrm{L})}}{\partial z} = -\frac{1}{c}\frac{\partial V^{(\mathrm{L})}}{\partial t} = ikV, \qquad V^{(\mathrm{L})} = -\frac{i}{k}\frac{\partial A_z^{(\mathrm{L})}}{\partial z}, \tag{33}$$

and the electric and magnetic fields follow from the potentials as,

$$\mathbf{E} = -\boldsymbol{\nabla} V^{(\mathrm{L})} - \frac{1}{c}\frac{\partial \mathbf{A}^{(\mathrm{L})}}{\partial t} = \frac{i}{k}\boldsymbol{\nabla}\frac{\partial A_z^{(\mathrm{L})}}{\partial z} + ikA_z^{(\mathrm{L})}\,\hat{\mathbf{z}}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{L})}, \tag{34}$$

$$E_x = \frac{i}{k}\frac{\partial^2 A_z^{(\mathrm{L})}}{\partial x \partial z}, \qquad E_y = \frac{i}{k}\frac{\partial^2 A_z^{(\mathrm{L})}}{\partial y \partial z}, \qquad E_z = \frac{i}{k}\left(\frac{\partial^2 A_z^{(\mathrm{L})}}{\partial z^2} + k^2 A_z^{(\mathrm{L})}\right), \tag{35}$$

$$B_x = \frac{\partial A_z^{(\mathrm{L})}}{\partial y}, \qquad B_y = -\frac{\partial A_z^{(\mathrm{L})}}{\partial x}, \qquad B_z = 0. \tag{36}$$

That is $k_z = 0$ for these modes, all of which have mode index $n = 0$.

It seems reasonable to complete the solution by enforcing the boundary conditions that the electric field must be normal, and the magnetic field must be tangential, to the perfectly conducting walls of the cavity. Note, however, that these boundary conditions involve combinations of derivatives of $V$ and components of $\mathbf{A}$, so it is not obvious that the solution for the potentials obtained using them will be unique, even if the $\mathbf{E}$ and $\mathbf{B}$ are satisfactory.

For $k_z = 0$, the potentials (inside the cavity) are,

$$A_z^{(\mathrm{L})} = -\frac{iE_0}{\sqrt{k_x^2 + k_y^2}} \sin k_x x \, \sin k_y y, \qquad V^{(\mathrm{L})} = 0, \tag{37}$$

which are the same as the Hamiltonian-gauge potentials (15) in this case.[9]

However, with $k_z = 0$, the surface charge density $\sigma = \pm E_z/4\pi$ has opposite signs on the walls at $z = 0$ and $z = d_z$, and is independent of $d_z$, such that the retarded potential $V^{(\mathrm{L,ret})}$ of eq. (29) is nonzero. For example, consider a high mode with odd indices $l$ and $m$ of a

[9]For completeness, we note that the unit vectors $\hat{\mathbf{e}}$ and $\hat{\mathbf{b}}$ have components,

$$e_x = -\frac{k_x k_z}{k\sqrt{k_x^2 + k_y^2}}, \qquad e_y = -\frac{k_y k_z}{k\sqrt{k_x^2 + k_y^2}}, \qquad e_z = \frac{\sqrt{k_x^2 + k_y^2}}{k},$$

$$b_x = \frac{k_y}{\sqrt{k_x^2 + k_y^2}}, \qquad b_y = -\frac{k_x}{\sqrt{k_x^2 + k_y^2}}, \qquad b_z = 0. \tag{38}$$

long, thin cavity where $kd_z = \sqrt{k_x^2 + k_y^2}\, d_z \gg 1$ and $d_x, d_y \ll d_z$. Then, for a point inside the cavity far from both ends ($z \gg 1/k$ and $d_z - z \gg 1/k$) the retarded potential (29) is,

$$V^{(\mathrm{L,ret})}(\mathbf{x}) \approx Q \left( \frac{e^{ikz}}{z} - \frac{e^{ik(d_z - z)}}{d_z - z} \right) \neq 0, \tag{39}$$

where $Q = E_0 d_x d_y / \pi^2 lm$ is the peak charge on the face $z = 0$. This potential also holds for points outside the cavity such that $\sqrt{x^2 + y^2}$ is small compared to $z \gg 1/k$ and $d_z - z \gg 1/k$. Only on the plane $z = d_z/2$ does $V^{(\mathrm{L,ret})} = 0$. Then, the retarded Lorenz-gauge vector potential in this region follows as,

$$\begin{aligned} A_z^{(\mathrm{L,ret})}(\mathbf{x}) = -\frac{i}{k} \left( \frac{\partial V^{(\mathrm{L,ret})}}{\partial z} + E_z \right) \quad &\approx \quad -\frac{iQ}{k} \left[ \left( ik - \frac{1}{z} \right) \frac{e^{ikz}}{z} + \left( ik - \frac{1}{d_z - z} \right) \frac{e^{ik(d-z)}}{d_z - z} \right] \\ &\qquad + \begin{cases} -\frac{iE_0}{k} \sin k_x x \sin k_y y & \text{(inside)}, \\ 0 & \text{(outside)}. \end{cases} \end{aligned} \tag{40}$$

Similar approximations can be given for the (nonzero) retarded Lorenz-gauge potentials outside the cavity for $z \ll 1/k$ and $z - d_z \gg 1/k$.

Both sets of potentials, $V^{(\mathrm{L,ret})}$, $\mathbf{A}^{(\mathrm{L,ret})}$ and $V^{(\mathrm{L})} = 0$, $\mathbf{A}^{(\mathrm{L})}$, satisfy the Lorenz-gauge condition (21), and so are related by a restricted gauge transformation as in eq. (28).

The potentials (37) were used in sec. 14.2 of [4] to compute the strength of excitation of the fundamental cavity mode by a passing charged particle.[10]

### 2.2.4 Coulomb Gauge

The Coulomb gauge (favored by Maxwell) is defined by the condition that $\boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{C})} = 0$. The Coulomb gauge (like the Lorenz gauge) is a velocity gauge, with the speed $v$ of propagation of the scalar potential $V^{(\mathrm{C})}$ being infinite. Coulomb-gauge potentials are not unique, but different variants can be related by a gauge transformations whose gauge function $\chi$ obeys $\nabla^2 \chi = 0$.

Coulomb-gauge potentials for cavities and waveguides are extensively discussed in chap. 13 of [12], to which the reader is referred.[11]

# A Appendix: Patch Antenna "Cavity"

A "patch" antenna consists of a rectangular (or circular, *etc.*) conductor (the "patch") above a more-or-less infinite conducting plane. The gap between these conductors is filled with a dielectric whose thickness is typically smaller than the diagonal of the rectangle.

[10] See also [11], which deduces the excitation without use of potentials.

[11] Smythe's discussion of cavity modes is based on prior discussion of waveguide modes, noting that standing waveguide modes are equivalent to a sequence of cavity modes. His discussion of waveguide modes is based on representation of the vector potential by so-called Hertz vectors and scalars together with the unit vector $\mathbf{k} = \hat{\mathbf{z}}$, taking the axis if the waveguides to be parallel to the $z$-axis.

An approximate analysis (see, for example, sec. 14.2.2 of [13]) of these devices supposes them to be a kind of rf cavity whose sides are "magnetic conductors" (which support no tangential magnetic field).The cavity model (see below) yields expressions for the electric currents on the "top" and "bottom", and the "magnetic currents" on the "sides," from which the far-zone radiation pattern can be deduced.

Of interest here is that the fields of this "cavity" can be deduced from a vector potential that obeys eq. (31, having the general form (32). For simplicity, we assume that the (relative) permittivity of the dielectric space is unity.

We take the "top" and "bottom" of the cavity to be the surfaces at $z = 0$ and $d_z$, such that the $z$-direction has a different physical significance to the $x$- and $y$-directions. This suggests that we seek solutions where the vector potential has only a $z$-component.

Of course, the scalar potential $V$ in the Lorenz gauge is related to the vector potential by,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{1}{c}\frac{\partial V}{\partial t} = ikV, \qquad V = -\frac{i}{k}\frac{\partial A_z}{\partial z}. \tag{41}$$

The electric and magnetic fields follow from the potentials according to eqs. (34)-(36). The boundary conditions are that the tangential electric field vanish on the electrical conductors,

$$E_x(x, y, 0) = E_x(x, y, d_z) = E_y(x, y, 0) = E_y(x, y, d_z) = 0, \tag{42}$$

and that the tangential magnetic field vanish on the "magnetic conductors,"

$$\begin{aligned} B_y(0, y, z) = B_y(d_x, y, z) = B_z(0, y, z) = B_z(d_x, y, z) = 0, \\ B_x(x, 0, z) = B_x(x, d_y, z) = B_z(x, 0, z) = B_z(x, d_y, z) = 0. \end{aligned} \tag{43}$$

The satisfactory form of eq. (32) inside the "cavity" is,[12]

$$A_z = \frac{E_0}{i\sqrt{k_x^2 + k_y^2}} \cos k_x x \, \cos k_y y \, \cos k_z z, \tag{44}$$

and the fields there are,

$$\begin{aligned}
E_x &= E_0 \frac{k_x k_z}{k\sqrt{k_x^2 + k_y^2}} \sin k_x x \, \cos k_y y \, \sin k_z z, \\
E_y &= E_0 \frac{k_y k_z}{k\sqrt{k_x^2 + k_y^2}} \cos k_x x \, \sin k_y y \, \sin k_z z, \\
E_z &= E_0 \frac{\sqrt{k_x^2 + k_y^2}}{k} \cos k_x x \, \cos k_y y \, \cos k_z z, \qquad (45) \\
B_x &= iE_0 \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \cos k_x x \, \sin k_y y \, \cos k_z z, \\
B_y &= -iE_0 \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \sin k_x x \, \cos k_y y \, \cos k_z z, \\
B_z &= 0. \qquad (46)
\end{aligned}$$

[12]As in sec. 2.2.3, the potential (44 is not the retarded potential, although it is a Lorenz-gauge potential.

The scalar potential inside the "cavity" is,

$$V = E_0 \frac{k_z}{k\sqrt{k_x^2 + k_y^2}} \cos k_x x \, \cos k_y y \, \cos k_z z. \tag{47}$$

The mode with indices $(l, m, n) = (0, 0, 0)$ has zero frequency, zero magnetic field, and corresponds to the electric field of the "cavity" considered as a DC capacitor. Supposing that $d_x > d_y > d_z$, the lowest-frequency mode can be labeled $\mathrm{TM}^z_{100}$ (in that the magnetic field of all modes from potential (44) is perpendicular to the $z$-axis).

Both **E** and **B** have nonzero tangential components on the "sides" of the "cavity", so the fields and the potentials are nonzero outside the "cavity." Indeed, the purpose of the "cavity" is to radiate energy. In the present approximation, **E** and **B** are 90° out of phase inside the cavity, so the time-average Poynting vector $\langle \mathbf{S} \rangle = cRe(\mathbf{E} \times \mathbf{B}^\star)/8\pi$ is zero inside the "cavity," including at its surface, so it is not obvious that the "cavity" radiates. However, further approximations, taking into account that $z = 0$ is effectively an infinite conducting plane, lead to useful estimates of the radiation [13, 14].

For completeness, we note that all modes of the patch antenna "cavity" are of the form,

$$\begin{aligned}
E_x &= E_0\, e_x \sin k_x x \, \cos k_y y \, \sin k_z z, \\
E_y &= E_0\, e_y \cos k_x x \, \sin k_y y \, \sin k_z z, \\
E_z &= E_0\, e_z \cos k_x x \, \cos k_y y \, \cos k_z z, && (48) \\
B_x &= iE_0\, b_x \cos k_x x \, \sin k_y y \, \cos k_z z, \\
B_y &= iE_0\, b_y \sin k_x x \, \cos k_y y \, \cos k_z z, \\
B_z &= iE_0\, b_z \sin k_x x \, \sin k_y y \, \sin k_z z. && (49)
\end{aligned}$$

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