Force between Two Comoving Electric Charges

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1 Problem

What is the electromagnetic force between two electric charges q that move with constant velocity v perpendicular to their line of centers? The separation between the charges is somehow held fixed by a "nonelectromagnetic" force.

2 Solution

This problem was briefly considered by FitzGerald in 1882 [1], as a prelude to discussion of the force on a moving electric charge due to a magnet that has the same velocity.[1] FitzGerald showed that the force is zero in this case, based on arguments in Arts. 598-599 of Maxwell's *Treatise* [3] (although FitzGerald did not specifically cite this). These lab-frame arguments were a precursor to a simpler, relativistic argument that in the rest frame of a charge and comoving magnetic there would be no magnetic force on the charge, and hence the lab-frame force is also zero.[2]

The first paragraph of FitzGerald's paper reads: *Professor Rowland has shown experimentally that a quantity of electricity moving acts like an electric current. This had been assumed by many physicists. It follows that two quantities of electricity moving in the same direction with the velocity of light, would have no action on one another, their electrostatic action being balanced by an equal and opposite electrokinetic action. As it is very unlikely that anything depends on absolute motion, the motion here spoken of must be with respect to something, and this can hardly be any other thing than the ether in space.*

FitzGerald did not support his claim, which we now examine from a more "relativistic" perspective.

We consider a pair of electric charges q whose separation is $\mathbf{d}_0$ in their common rest frame, where the electric field of the first at the position of the second is (in Gaussian units)

$$\mathbf{E}_0 = \frac{q}{d_0^2}\,\hat{\mathbf{d}}_0, \tag{1}$$

and the electrical force on the second charge is,

$$\mathbf{F}_0 = q\mathbf{E}_0 = \frac{q^2}{d_0^2}\,\hat{\mathbf{d}}_0. \tag{2}$$

[1]FitzGerald's paper was a comment on an important work of Rowland [2] which provided the first experimental evidence that moving electric charges have a magnetic field.

[2]For more discussion of Maxwell and special relativity, see [4].

2.1 $\mathbf{v} \perp \mathbf{d}_0$

When the charges are in motion with velocity $\mathbf{v}$ perpendicular to the line of centers $\mathbf{d}_0$, the electric field of the first at the position of the second is (sec. 6 of [5]),

$$\mathbf{E} = \gamma \mathbf{E}_0 \qquad (\mathbf{v} \perp \mathbf{d}_0), \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,, \tag{3}$$

and the lab-frame magnetic field at this point is[3]

$$\mathbf{B} = \frac{\mathbf{v}}{c} \times \mathbf{E} = \gamma \frac{\mathbf{v}}{c} \times \mathbf{E}_0 \qquad (\mathbf{v} \perp \mathbf{d}_0). \tag{4}$$

The lab-frame Lorentz force on the second charge is then, noting that $\mathbf{d} \cdot \mathbf{v} = 0$,

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = q\mathbf{E}\left(1 - \frac{v^2}{c^2}\right) = \frac{q\mathbf{E}}{\gamma^2} = \frac{q\mathbf{E}_0}{\gamma} \qquad (\mathbf{v} \perp \mathbf{d}_0). \tag{5}$$

As v approaches c, $\gamma \to \infty$, and $\mathbf{F} \to 0$.

As Rowland showed [2], a moving electric charge is a kind of magnet, and the (lab-frame) magnetic force of a moving charge on a comoving electric charge tends to cancel the (lab-frame) electric force between them, when the velocity $\mathbf{v}$ is perpendicular to the line of centers of the charges. In the limit that $v \to c$, this cancelation is complete.

2.2 $\mathbf{v} \parallel \mathbf{d}_0$

However, if the velocity $\mathbf{v}$ is parallel to the line of centers of the charges, then the lab-frame electric field of the first at the position of the second is,

$$\mathbf{E} = \mathbf{E}_0 = \frac{q}{d_0^2}\,\hat{\mathbf{d}}_0 = \gamma \frac{q}{d^2}\,\hat{\mathbf{d}} \qquad (\mathbf{v} \parallel \mathbf{d}_0), \tag{6}$$

where the lab-frame separation of the charges is $\mathbf{d} = \mathbf{d}_0/\gamma$ due to the Lorentz contraction, while the lab-frame magnetic field at the second charge is zero, according to the first form of eq. (4). Then, the lab-frame force on the second charge is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = q\mathbf{E} = q\mathbf{E}_0 = \gamma^2 \frac{q^2}{d^2}\,\hat{\mathbf{d}} \qquad (\mathbf{v} \parallel \mathbf{d}_0), \tag{7}$$

which diverges as $v \to c$.

2.3 Comments

Thus, the lab-frame force goes to zero as $v \to c$ only if $\mathbf{v}$ is perpendicular to the line of centers $\mathbf{d}_0$ of the charges, while FitzGerald's claim seems to be that the force would be independent of $\mathbf{v} \cdot \mathbf{d}_0$.[4]

[3] The first form of eq. (4) was first explicitly given in [6] for $v \ll c$ (although implicit in secs. 769-770 of [3]; see also [4]), and argued to be true for uniform motion with any v in [7, 8].

[4] J.J. Thomson made a similar claim on p. 248 of [6], where he argued that the force goes to zero as $v \to \sqrt{3}\,c$. Thomson contrasted his result, which he believed to follow from Maxwell's equations, with Weber's force law [9], $\mathbf{F} = (qq'\,\hat{\mathbf{d}}/d^2)(1 - \dot{d}^2/2c^2 + d\ddot{d}/c^2)$, where $\mathbf{d}$ is the separation of charges q and q', which force would be just $qq'\,\hat{\mathbf{d}}/d^2$ for comoving charges, independent of their velocity $\mathbf{v}$.

Also, the "relativistic" view is that the results (5) and (7) hold in any inertial (lab) frame in which the charges have velocity **v**, while FitzGerald implied that eq. (5) would hold only if the charges had velocity **v** with respect to the "ether". Yet, FitzGerald's 1882 paper [1] was one of the first indications that the speed of light plays a role as a limiting velocity for particles.

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