Electron Trajectories in a Vacuum Coaxial Cable

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1 Problem

A coaxial cable has inner conductor of radius a, outer conductor of radius b, and vacuum between. A constant voltage V is maintained between the conductors, and steady current I flows on the inner conductor (with current $-I$ on the outer conductor). Electrons leave the inner conductor with negligible velocity (due to thermionic emission) and are attracted to the outer conductor. Show that the electrons cannot reach the outer conductor if

$$I > \frac{cV}{2\ln b/a}\sqrt{1 + \frac{2mc^2}{eV}}, \tag{1}$$

in Gaussian units, where $e > 0$ and m are the charge and mass of the electron and c is the speed of light. The fields due to the electrons in the vacuum can be ignored.

This problem was first considered by Richardson [1], who gave only a nonrelativistic analysis. See also [2]. A relativistic analysis adds the term 1 in eq. (1), which is negligible compared to mc^2/eV in most practical cases.

The problem can be solved in the lab frame, or by transforming to a moving frame in which one of $\mathbf{E}$ *or* $\mathbf{B}$ *is zero.*

2 Solution

The electric field $\mathbf{E}$ between the two conductors is radial with magnitude

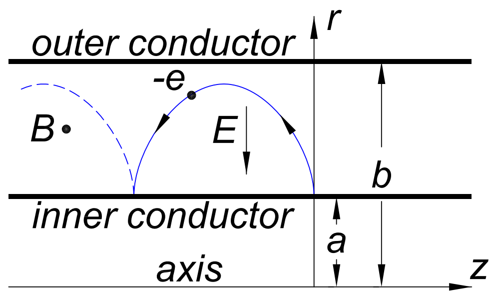
$$E_r(r) = -\frac{V}{r\ln b/a}, \tag{2}$$

and the magnetic field $\mathbf{B}$ is azimuthal with magnitude

$$B_\phi(r) = \frac{2I}{cr}, \tag{3}$$

in a cylindrical coordinate system (r, ϕ, z).

The Lorentz force on an electron is $-e(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$. An electron emitted with zero velocity at $r = a$ first accelerates radially, but once it acquires a radial velocity v_r the interaction $-e\mathbf{v}/c \times \mathbf{B} = -(ev_r B_\phi/c)\,\hat{\mathbf{z}}$ leads to a (negative) axial component v_z of the velocity as well. Then, the interaction of the axial velocity with the azimuthal magnetic field leads to a (negative) radial force $ev_z B_\phi/c < 0$ that opposes the radial force $-eE_r > 0$ due to the electric field. The trajectory of the electron lies in a plane of constant azimuth (an r-z plane) and has the cycloidal form sketched on the following page, supposing the electron does not reach radius b.



The energy of the electron is only affected by the electric field, so the total (relativistic) energy U of the electron at radius r is

$$U(r) = \gamma(r)mc^2 = mc^2 + eV(r) = mc^2 + eV\frac{\ln r/a}{\ln b/a}\,, \tag{4}$$

where

$$\gamma(r) = \frac{1}{\sqrt{1 - v^2(r)/c^2}} = 1 + \frac{eV}{mc^2}\frac{\ln r/a}{\ln b/a}\,. \tag{5}$$

Assuming the electron is emitted with zero velocity at radius $r = a$ we can ignore azimuthal motion, and the equation of motion of the electron can be written

$$\frac{d\mathbf{P}}{dt} = m\frac{d}{dt}(\gamma\mathbf{v}) = -e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{6}$$

whose r and z components are,

$$m\frac{d}{dt}(\gamma\dot{r}) = -eE_r + \frac{eB_\phi}{c}\dot{z} = \frac{eV}{r\ln b/a} + \frac{2eI}{c^2}\frac{\dot{z}}{r}\,, \tag{7}$$

where $\dot{r} = dr/dt$, and

$$m\frac{d}{dt}(\gamma\dot{z}) = -\frac{eB_\phi}{c}\dot{r} = -\frac{2eI}{c^2}\frac{\dot{r}}{r}\,. \tag{8}$$

Equation (8) can be integrated to give

$$\dot{z} = -\frac{2eI}{\gamma mc^2}\ln\frac{r}{a}\,, \tag{9}$$

recalling that $\dot{z} = 0$ when $r = a$. Multiplying eq. (7) by $\gamma\dot{r}$ and using eq. (9), we have

$$\gamma\dot{r}\frac{d}{dt}(\gamma\dot{r}) = \frac{eV}{m\ln b/a}\frac{\gamma\dot{r}}{r} - \left(\frac{2eI}{mc^2}\right)^2\ln\frac{r}{a}\frac{\dot{r}}{r}\,. \tag{10}$$

From eq. (5) we have that

$$\dot{\gamma} = \frac{eV}{mc^2\ln b/a}\frac{\dot{r}}{r}\,, \tag{11}$$

so eq. (10) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}(\gamma\dot{r})^2 = \frac{c^2}{2}\frac{d\gamma^2}{dt} - \left(\frac{2eI}{mc^2}\right)^2\ln\frac{r}{a}\frac{\dot{r}}{r}\,, \tag{12}$$

which integrates to

$$\frac{\gamma^2 \dot{r}^2}{2} = \frac{c^2}{2}(\gamma^2 - 1) - \frac{1}{2}\left(\frac{2eI}{mc^2}\right)^2 \ln^2 \frac{r}{a}\,, \tag{13}$$

noting that $\dot{r} = 0$ at $r = a$.

If the electron's trajectory reaches a maximum radius of $r = b$, then $\dot{r} = 0$ there. Thus, we can identify the critical current by setting eq. (13) to zero at $r = b$, which leads to

$$I_{\rm crit} = \frac{mc^3}{2e \ln b/a}\sqrt{\gamma^2(b) - 1} = \frac{cV}{2 \ln b/a}\sqrt{1 + \frac{2mc^2}{eV}}, \tag{14}$$

recalling eq. (5).

2.1 Analysis for $|E| < |B|$ in a Moving Frame

For an alternative analysis we follow the hint by transforming to the $'$ frame that has velocity $\mathbf{u}$ along the z axis with the goal that either $\mathbf{E}' = 0$ or $\mathbf{B}' = 0$. Comparing eqs. (2)-(3) with eq. (1) we see that the magnetic field is larger than the electric field when the electrons cannot reach $r = b$. In this case we can transform to a frame in which the electric field vanishes by taking

$$\mathbf{u} = c\frac{\mathbf{E} \times \mathbf{B}}{B^2} = c\frac{E_r}{B_\phi}\,\hat{\mathbf{z}} = -\frac{c^2 V}{2I \ln b/a}\,\hat{\mathbf{z}}. \tag{15}$$

Then in the $'$ frame the fields are

$$E'_r = \gamma_u(E_r - u_z B_\phi/c) = 0, \qquad B'_\phi = \gamma_u(B_\phi - u_z E_r/c) = \frac{B_\phi}{\gamma_u}\,, \tag{16}$$

where

$$\gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - E_r^2/B_\phi^2}}\,. \tag{17}$$

In the $'$ frame an electron has velocity $-\mathbf{u}$ when it leaves the conductor at $r = a$. Because only a magnetic field exists in this frame, the magnitude of the electron's velocity remains constant along its trajectory, and the γ-factor of the electron is just γ_u of eq. (17).

The equation of motion in the $'$ frame is

$$\frac{d\mathbf{P}'}{dt'} = \gamma_u m \frac{d\mathbf{v}'}{dt'} \equiv \gamma_u m \dot{\mathbf{v}}' = -e\frac{\mathbf{v}'}{c} \times \mathbf{B}', \tag{18}$$

whose r' and z' components are, noting that $r' = r$,

$$\gamma_u m \ddot{r}' = \frac{eB_\phi}{\gamma_u c}\dot{z}' = \frac{2eI}{\gamma_u c^2}\frac{\dot{z}'}{r'}\,, \tag{19}$$

and

$$\gamma_u m \ddot{z}' = -\frac{eB_\phi}{\gamma_u c}\dot{r}' = -\frac{2eI}{\gamma_u c^2}\frac{\dot{r}'}{r'}\,. \tag{20}$$

Equation (20) can be integrated to give

$$\dot{z}' = -\frac{2eI}{\gamma_u^2 mc^2} \ln \frac{r'}{a} - u_z, \tag{21}$$

recalling that $\dot{z} = -u_z > 0$ when $r = r' = a$. Multiplying eq. (19) by $\dot{r}'$ and using eq. (21), we have

$$\dot{r}'\ddot{r}' = -\frac{2eI}{\gamma_u^2 mc^2} \left(\frac{2eI}{\gamma_u^2 mc^2} \ln \frac{r'}{a} \frac{\dot{r}'}{r'} + u_z \frac{\dot{r}'}{r'} \right), \tag{22}$$

which integrates to

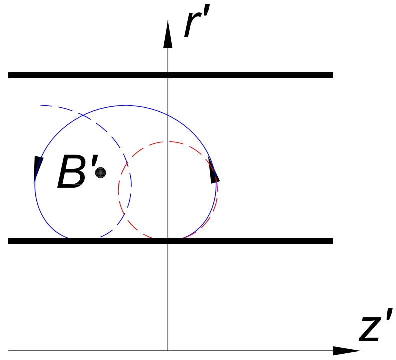
$$\frac{\dot{r}'^2}{2} = -\frac{2eI}{\gamma_u^2 mc^2} \left(\frac{eI}{\gamma_u^2 mc^2} \ln^2 \frac{r'}{a} + u_z \ln \frac{r'}{a} \right) = -\frac{2eI}{\gamma_u^2 mc^2} \ln \frac{r'}{a} \left(\frac{eI}{\gamma_u^2 mc^2} \ln \frac{r'}{a} + u_z \right), \tag{23}$$

noting that $\dot{r}' = 0$ at $r' = a$.

Equation (23) holds for all cases when $|E_r| < |B_\phi|$. In these cases there is always some radius b for which $\dot{r}' = 0$, and the electron's trajectory is limited to $r = r' \le b$. Setting eq. (23) to zero and recalling eqs. (15) and (17), we find after some rearrangement that the critical current is

$$I_{\rm crit} = \frac{cV}{2 \ln b/a} \sqrt{1 + \frac{2mc^2}{eV}}. \tag{24}$$

Comparing eqs. (21) and (24), we see that $\dot{z}' = u_z < 0$ when $r' = b$ although $\dot{z}' = -u_z > 0$ at $r' = a$. Hence, $\dot{z}' = 0$ at some radius c where $a < c < b$. Recall that if the magnetic field were uniform, the electron's trajectory would be a circle with radius of curvature $\rho = cP/eB$, where $P = \gamma_u mu$ is the electron's momentum. Since $B \propto 1/r = 1/r'$ in the present example, the radius of curvature of the electron's trajectory in the $'$ frame scale with r', and the trajectory has the form of the blue curve (based on numerical integration of eqs. (19)-(20)) sketched below for $I > I_{\rm crit}$.



Because the electron's trajectory drifts to negative z in the $'$ frame, the average drift velocity in the lab frame is larger in magnitude than that of u_z of eq. (15).

If the fields $\mathbf{E}$ *and* $\mathbf{B}$ *were uniform, then* $\mathbf{B}'$ *would be uniform also, and the electron's trajectory in the* $'$ *frame would be a circle with center at* $(r' > a, 0, 0)$ *that passes through the point* $(r' = a, 0, 0)$*. In the figure above the red, dashed curve would be the circular orbit if the fields were uniform with the values of eqs. (2)-(3) at* $r = a$*. The average velocity of the electron in the* $'$ *frame would be zero, so the average velocity in the z direction in the lab frame would be* $\mathbf{u}$ *of eq. (15), which is the well-known* $E \times B$ *drift velocity (see, for example, sec. 22 of [3] or sec. 12.3 of [4]).*

2.2 $|E| > |B|$

If the electric field is stronger than the magnetic field the preceding analysis does not hold. In this case we can transform to the $'$ frame with velocity

$$\mathbf{u} = c\frac{\mathbf{E}\times\mathbf{B}}{E^2} = c\frac{B_\phi}{E_r}\,\hat{\mathbf{z}} = -\frac{2I\ln b/a}{V}\,\hat{\mathbf{z}}, \tag{25}$$

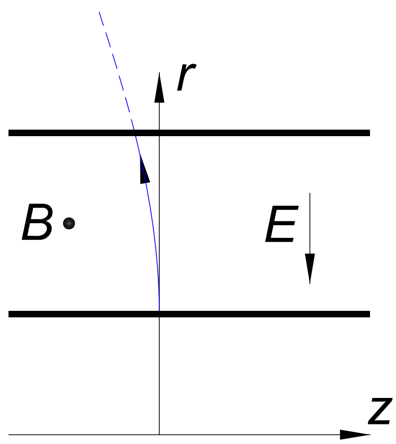
in which frame the fields are

$$E'_r = \gamma_u(E_r - u_z B_\phi/c) = \frac{E_r}{\gamma_u} \qquad B'_\phi = \gamma_u(B_\phi - u_z E_r/c) = 0, \tag{26}$$

where

$$\gamma_u = \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-B_\phi^2/E_r^2}}\,. \tag{27}$$

In the $'$ frame there is only a radial electric field, so the electrons move radially without bound (until they strike the outer conductor). In the lab frame, the electron's trajectory is nearly radial, with a slight deflection to negative z, as sketched below.



2.3 Constants of Motion via a Lagrangian Analysis

It may be of interest to record some aspects of a Lagrangian analysis of this problem.

The Lagrangian function $\mathcal{L}$ for an electron of charge $-e$ that interacts with electromagnetic fields with scalar potential V and vector potential $\mathbf{A}$ is (see, for example, sec. 16 of [3] or sec. 12.1 of [4])

$$\mathcal{L} = -mc^2\sqrt{1-v^2/c^2} + eV - \frac{e}{c}\mathbf{v}\cdot\mathbf{A}. \tag{28}$$

In the present example the potentials in cylindrical coordinates (r, ϕ, z) are

$$V(r) = V\frac{\ln r/a}{\ln b/a}\,, \qquad \text{and} \qquad \mathbf{A}(r) = -\frac{2I}{c}\ln\frac{r}{a}\,, \tag{29}$$

in the region $a \le r \le b$, where they are independent of coordinates ϕ and z. Hence, there are two conserved canonical momenta,

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \gamma m r^2\dot\phi - \frac{er}{c}A_\phi = \gamma m r^2\dot\phi = rP_\phi = L_z, \tag{30}$$

where $P_\phi = \gamma m r\dot\phi$ is the ϕ component of mechanical momentum $\mathbf{P}$, and

$$p_z = \frac{\partial \mathcal{L}}{\partial \dot z} = \gamma m \dot z + \frac{2eI}{c^2}\ln\frac{r}{a}. \qquad (31)$$

This first of these, p_ϕ, is the (canonical) angular momentum (about the z axis) L_z, which vanishes for an electron emitted from rest at $r = a$, so that $\dot\phi = 0$ at all times and the electron's trajectory lies in a plane of constant azimuth ϕ, as noted above. The other conserved canonical momentum, p_z, also vanishes for an electron emitted from rest, so we again obtain eq. (9) for the axial velocity $\dot z$.

The Lagrangian is independent of time in the present example, so the Hamiltonian $\mathcal{H}$ is constant,

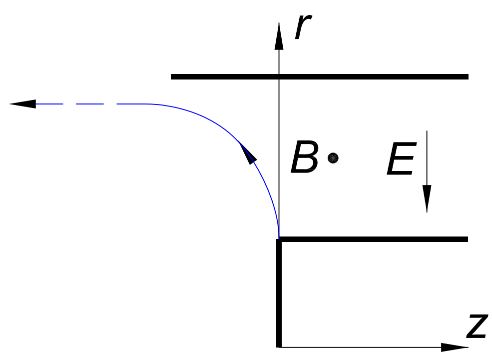
$$\mathcal{H} = \mathbf{v}\cdot\frac{\partial \mathcal{L}}{\mathbf{v}} - \mathcal{L} = \gamma m c^2 - eV. \qquad (32)$$

For an electron emitted from rest at $r = a$ where $V = 0$ we again obtain eq. (4).

3 Lorentz Force Accelerators/MPD Thrusters

This comment added Feb. 2008.

The principles of this problem underlie so-called Lorentz force accelerators = magnetoplasmadynamic (MPD) thrusters [5]. Electrons (or ions) are accelerated radially by the electric field, and their radial momentum is converted into longitudinal momentum by the azimuthal magnetic field. The longitudinal extent of the latter is chosen so the particles are deflected by 90°, such that the reaction force on the currents that generate the magnetic field provides a thrust in the $+z$ direction.



Of course, the $\mathbf{v}\times\mathbf{B}$ Lorentz force does not directly accelerate any particles in the sense of changing their energy. The Lorentz force changes the direction of an electron's velocity but not its magnitude. However, the reaction to the Lorentz force can change the velocity of the system that generates the magnetic field (if this system is free to move).

References

[1] O.W. Richardson, *The Origin of Thermal Ionisation from Carbon*, Proc. Roy. Soc. London **A90**, 174 (1914),
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[2] A.W. Hull, *The Motion of Electrons between Coaxial Cylinders under the Influence of a Current along the Axis*, Phys. Rev. **25**, 645 (1945),
http://physics.princeton.edu/~mcdonald/examples/EM/hull_pr_25_645_25.pdf

[3] L.D. Landau and E.M. Lifshitz, *Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, Oxford, 1987).

[4] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).

[5] A sample of the vast literature on MPD's is M.R. LaPointe and P.G. Mikellides, *High Power MPD Thruster Development at the NASA Glenn Research Center*, NASA/CR-2001-211114, AIAA20013499 (2001),
http://physics.princeton.edu/~mcdonald/examples/accel/lapointe_nasa-cr-2001-211114.pdf