A Phased Antenna Array

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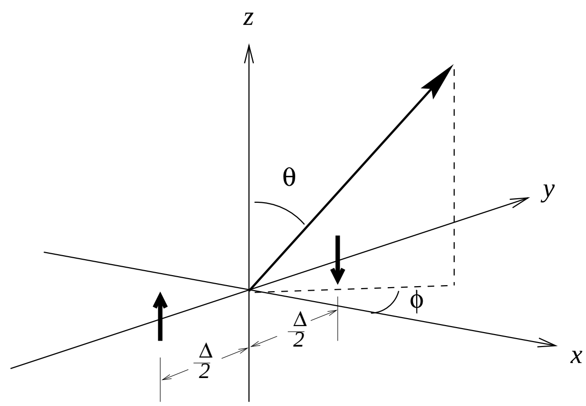
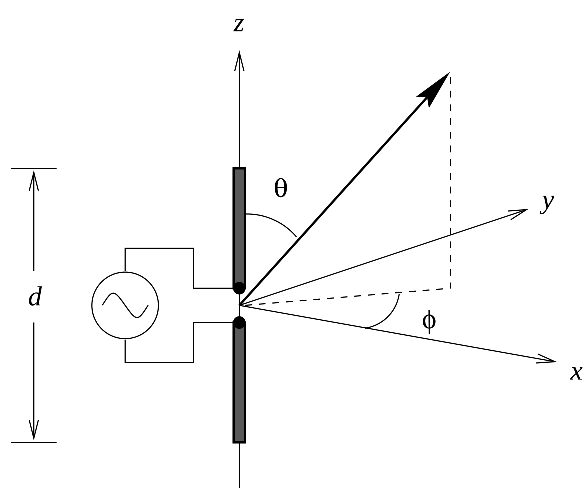
1 Problem

Two "short" dipole antennas form a small "phased array" as shown in the figure. The second dipole is placed a distance $\Delta = \lambda/2$ away from the first along the y axis. The two dipoles are parallel to one another and are driven 180° out of phase of one another.



Each antenna is a center-fed dipole radiator formed from two wires, each of length $d/2 \ll \lambda$ and driven by a current source as shown in the figure below. The wires are aligned parallel to the z axis ($\theta = (0, \pi)$). The current source produces a time-dependent current given by $I(t) = I_0 e^{-i\omega t}$. You may assume that the charge that enters the wires is uniformly distributed along their lengths.



Calculate the time averaged angular distribution of the radiated power for this arrangement in the radiation zone as a function of θ and ϕ, *i.e.*, calculate $\langle dP(\theta, \phi)/d\Omega \rangle$.

Consider also the case that the two antennas are driven 90° out of phase, and their separation is only $\lambda/4$.

2 Solution

2.1 Radiation Pattern

The amplitude A_0 of the radiation from a single "short" antenna varies with angle as $\sin\theta$. The (time-averaged) radiated power has angular distribution

$$\frac{dP_1(\theta,\phi)}{d\Omega} = A_0^2 \sin^2\theta. \tag{1}$$

We first discuss the angular distribution of the two-antenna system, then return to the issue of the normalization of eq. (1).

Consider two antennas separated in space by vector distance $\mathbf{L}$, and operated with phase difference $\delta\varphi_0$ between them. When viewed by a distant observer along direction $\hat{\mathbf{n}}$, the path length difference of the radiation of the two antennas to the observer is $\hat{\mathbf{n}}\cdot\mathbf{L}$. The total phase difference between the radiation from the two antennas is therefore

$$\delta\varphi = 2\pi\frac{\hat{\mathbf{n}}\cdot\mathbf{L}}{\lambda} + \delta\varphi_0. \tag{2}$$

The total amplitude of the radiation from the two antennas is

$$A = A_0 \sin\theta(1 + e^{i\delta\varphi}) = 2A_0 e^{i\delta\varphi/2}\sin\theta\cos\frac{\delta\varphi}{2}. \tag{3}$$

The (time-averaged) radiated power from the two antennas is then

$$\frac{dP_2}{d\Omega} = \left|A^2\right| = 4A_0^2\sin^2\theta\cos^2\frac{\delta\varphi}{2} = 4A_0^2\sin^2\theta\cos^2\left(\pi\frac{\hat{\mathbf{n}}\cdot\mathbf{L}}{\lambda} + \frac{\delta\varphi_0}{2}\right). \tag{4}$$

In the present example, $\mathbf{L} = \Delta\hat{\mathbf{y}} = \lambda\hat{\mathbf{y}}/2$, $\delta\varphi_0 = \pi$, and, of course, $\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$, so that

$$\begin{aligned}
\frac{dP_2(\theta,\phi)}{d\Omega} &= 4A_0^2\sin^2\theta\cos^2\left(\frac{\varphi_0}{2} + \frac{\pi\Delta}{\lambda}\sin\theta\sin\phi\right) \\
&= 4A_0^2\sin^2\theta\left[\cos^2\frac{\delta\varphi_0}{2}\cos^2\left(\frac{\pi\Delta}{\lambda}\sin\theta\sin\phi\right) + \sin^2\frac{\delta\varphi_0}{2}\sin^2\left(\frac{\pi\Delta}{\lambda}\sin\theta\sin\phi\right)\right. \\
&\qquad\left. -\frac{1}{2}\sin\delta\varphi_0\sin\left(\frac{2\pi\Delta}{\lambda}\sin\theta\sin\phi\right)\right] \\
&= 4A_0^2\sin^2\theta\sin^2\left(\frac{\pi}{2}\sin\theta\sin\phi\right).
\end{aligned} \tag{5}$$

In the horizontal plane ($\theta = \pi/2$), the (time-averaged) radiation pattern is

$$\frac{dP_2(\pi/2,\phi)}{d\Omega} = 4A_0^2\sin^2\left(\frac{\pi}{2}\sin\phi\right). \tag{6}$$

This vanishes for $\phi = 0°,\ 180°$, and is maximal when $\phi = \pm 90°$. That is, the radiation is preferentially emitted along the y axis, the axis of the antenna array.

2.2 Normalization

The total, time-average radiated power from a single antenna follows from eq. (1) as

$$P_1 = \int \frac{dP_1}{d\Omega}\, d\Omega = A_0^2 \int \sin^2\theta\, d\Omega = \frac{8\pi}{3} A_0^2 = 8.38 A_0^2 \equiv P_0. \tag{7}$$

The total, time-average radiated power from the two antennas follows from eq. (5) as

$$P_2 = \int \frac{dP_2}{d\Omega}\, d\Omega = 4A_0^2 \int \sin^2\theta \sin^2\left(\frac{\pi}{2}\sin\theta\sin\phi\right) d\Omega = 19.30 A_0^2 = 2.30 P_0, \tag{8}$$

where the integral was evaluated using WolframAlpha.

If the total power that could be delivered to the two antennas is just $2P_0$, then eq. (5) should be renormalized by the factor $2/2.30 = 0.87$. The need for such a factor indicates that the two antennas interact with one another, and the radiation of one affects the behavior of the other. This effect is said to be due to the mutual inductance of the two antennas (which quantity won't be pursued further here).[1]

We do consider a related issue: Supposing the total drive power available is P_0, what is the improvement in the power radiated into the most favorable direction by the use of two antennas compared to just one?

From eq. (1), the maximum radiated power occurs for $\theta = \pi/2$,

$$\frac{dP_{1,\max}}{d\Omega} = A_0^2. \tag{9}$$

If the total drive power for both antennas is only P_0, eq. (5) must be renormalized by 1/2.30 = 0.43, Then, the peak radiated power, at $\theta = \pi/2$, $\phi = \pm\pi/2$ is[2]

$$\frac{dP_{2,\max}}{d\Omega} = 1.74 A_0^2 = 1.74 \frac{dP_{1,\max}}{d\Omega}\,. \tag{10}$$

2.3 Two Antennas with $\Delta = \lambda/4$, $\varphi_0 = \pi/2$

If $\mathbf{L} = \lambda\,\hat{\mathbf{y}}/4$ and $\varphi_0 = \pi/2$, the time-average radiation pattern of the two antennas follows from eq. (5) as

$$\frac{dP_2(\theta,\phi)}{d\Omega} = \; = \; 2A_0^2 \sin^2\theta \left[1 - \sin\left(\frac{\pi}{2}\sin\theta\sin\phi\right)\right]. \tag{11}$$

The total, time-average radiated power from the two antennas follows from eq. (11) as

$$P_2 = \int \frac{dP_2}{d\Omega}\, d\Omega = \frac{16\pi}{3} A_0^2 = 2P_0. \tag{12}$$

[1]In case the sources are excited atoms or other unstable particles, if they are sufficiently close to one another the effect of their mutual interaction is to reduce the particles' lifetimes. See R.H. Dicke, *Coherence in Spontaneous Radiation Processes*, Phys. Rev. **93**, 99 (1954),
http://physics.princeton.edu/~mcdonald/examples/QED/dicke_pr_93_99_54.pdf

[2]Similarly, if the total drive power is $2P_0$, the renormalization factor is 2 / 2.30 = 0.87, and $dP_{2,\max}/d\Omega = 3.48\, dP_{1,\max}/d\Omega$.

In this case, there is no mutual interaction between the antennas, such that the power radiated by the coherent superposition of the two antennas equals the sum of their individual powers, which result also applies to incoherent superposition. As such, this famous case may lead to the incorrect impression that coherent superpostion can be applied without a renormalization factor, such that in general the power radiated by the pair of antennas is greater than their total drive power.

If the total available drive power is only P_0, then eq. (11) is renormalized by a factor of 1/2, and the peak radiated power occurs for $\theta = \pi/2$, $\phi = -\pi/2$, with

$$\frac{dP_{2,\max}}{d\Omega} = 2A_0^2 = 2\frac{dP_{1,\max}}{d\Omega}\,. \tag{13}$$

Appendix: Radiated Power *vs.* Terminal Current I_0

For completeness, we relate A_0^2 to the current I_0 at the antenna terminals by considerations of a single antenna. The total radiated power is given by an appropriate version of Larmor's formula (in Gaussian units):

$$P = \frac{2\ddot{p}^2}{3c^3} = \frac{2\omega^4 p^2}{3c^3}, \tag{14}$$

where p is the dipole moment and ω is the frequency. The time-averaged power is then

$$\langle P \rangle = \frac{\omega^4 p_0^2}{3c^3}, \tag{15}$$

The angular distribution varies as $\sin^2\theta$, and so must be

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^4 p_0^2 \sin^2\theta}{8\pi c^3}. \tag{16}$$

The hint is that the charge distribution $\rho(z,t) = \rho_0(z)e^{-i\omega t}$ is actually uniform in each wire of the antenna: $\rho_0(0 < z < d/2) = \rho_0 =$ constant. Of course, $\rho_0(-z) = -\rho_0(z)$. The dipole moment is

$$p_0 = \int_{-d/2}^{d/2} \rho z dz = \frac{\rho_0 d^2}{4}. \tag{17}$$

To get ρ_0, we must relate the charge to the current, which is usefully done via the continuity equation, $\nabla \cdot \mathbf{j} = -\dot{\rho}$. For our one-dimensional problem, this can be re-expressed as

$$\frac{\partial I}{\partial z} = -\dot{\rho} = i\omega\rho_0 e^{-i\omega t} \qquad (0 < z < d/2), \tag{18}$$

which integrates to

$$I(z,t) = C(t) + i\rho_0\omega z e^{-i\omega t} \qquad (0 < z < d/2). \tag{19}$$

Now, $I(d/2) = 0$, so $C(t) = -i\rho_0\omega(d/2)e^{-i\omega t}$, and

$$I(z,t) = -\frac{i\rho_0\omega d}{2}\left(1 - \frac{2z}{d}\right)e^{-i\omega t} \qquad (0 < z < d/2). \tag{20}$$

That is,

$$I_0 = -\frac{i\rho_0 \omega d}{2}, \qquad \text{and} \qquad \rho_0 = \frac{2I_0}{\omega d}. \tag{21}$$

The full expression for the current distribution is

$$I(z,t) = I_0 \left(1 - \frac{2\,|z|}{d}\right) e^{-i\omega t} \qquad (-d/2 < z < d/2). \tag{22}$$

Combining eqs. (1), (16), (17), and (21), and noting that $\omega/c = k = 2\pi/\lambda$, we have

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^2 d^2 I_0^2 \sin^2\theta}{32\pi c^3} = \frac{\pi I_0^2 \sin^2\theta}{8c} \left(\frac{d}{\lambda}\right)^2 \equiv A_0^2 \sin^2\theta. \tag{23}$$

Hence,

$$A_0^2 = \frac{\pi I_0^2}{8c} \left(\frac{d}{\lambda}\right)^2. \tag{24}$$

Inserting this in eq. (5), the time-averaged power radiated by the two antennas is

$$\left\langle \frac{dP(\theta,\phi)}{d\Omega} \right\rangle = \frac{\pi I_0^2}{2c} \left(\frac{d}{\lambda}\right)^2 \sin^2\theta \sin^2\left(\frac{\pi}{2}\sin\theta\sin\phi\right). \tag{25}$$