Electromagnetic Field Momentum

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1 From Poynting to Poincaré

This note is in response to a question by Romer [1].

A difficulty in interpreting Poynting's vector **S** [2] as proportional to momentum for a system that includes sources as well as fields was first pointed out by Poincaré [3]. A relativistically consistent formalism can only be achieved by adding terms that include stresses in the sources that arise when the fields are generated.

The usual relativistic argument begins by recasting the Lorentz-force-density 4-vector,

$$f_\mu = F_{\mu\nu} J^\nu = \left(\frac{\mathbf{J} \cdot \mathbf{E}}{c}, \rho \mathbf{E} + \frac{\mathbf{J}}{c} \times \mathbf{B} \right), \tag{1}$$

on a current density J^μ in electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ (with 4-tensor $F_{\mu\nu}$, in Gaussian units where c is the speed of light in vacuum, and with metric $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$; Greek indices run over 0,1,2,3 while Latin indices run over 1,2,3) as the derivative of a stress tensor:

$$f_\mu = -\frac{\partial T_{\mu\nu}}{\partial x_\nu} = -\partial^\nu T_{\mu\nu}. \tag{2}$$

This leads to the result

$$T_{\mu\nu} = \frac{1}{4\pi} F_{\mu\alpha} F_\nu^\alpha + \frac{1}{16\pi} \eta_{\mu\nu} F_\alpha^\beta F_\alpha^\beta = \frac{1}{4\pi} \begin{pmatrix} (E^2 + B^2)/2 & \mathbf{E} \times \mathbf{B} = \mathbf{S}/c \\ \mathbf{E} \times \mathbf{B} & \delta_{ij}(E^2 + B^2)/2 - E_i E_j - B_i B_j \end{pmatrix}. \tag{3}$$

Next, one makes a trial definition of an energy-momentum 4-vector for the fields as

$$P_\mu = \int T_{0\mu}\, d\mathrm{Vol}, \tag{4}$$

so that

$$P_0 = \int T_{00}\, d\mathrm{Vol} = \frac{1}{8\pi} \int (E^2 + B^2)\, d\mathrm{Vol} = U_f, \tag{5}$$

$$P_i = \int T_{0i}\, d\mathrm{Vol} = \frac{1}{4\pi} \int \mathbf{E} \times \mathbf{B}\, d\mathrm{Vol} = c\mathbf{P}_f, \tag{6}$$

where

$$\mathbf{P}_f = \frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{B}\, d\mathrm{Vol} \tag{7}$$

is the field 3-momentum that is the subject of Question #26 [1]. Then, $P_\mu = (U, c\mathbf{P}_f)$ has the appearance of a familiar 4-vector.

2 Free Fields

If there are no sources present (free-field case), then the Lorentz-force 4-vector vanishes, the 4-divergence of $T_{\mu\nu}$ vanishes also, and one can verify that P_μ really transforms like a 4-vector.

The argument thus far is seconded in the books of Rohrlich [4] and of Jackson [5], who don't carrying it much further.

3 Fields with Sources

Poincaré suggests we proceed to the case where sources of the fields are present. By direct application of a Lorentz transformation to the stress tensor $T^\star_{\mu\nu}$, where the $\star$ indicates the rest frame of the sources, one deduces that P_μ fails to transform like a 4-vector if there are nonzero spatial components to the stress tensor, *i.e.*, if some $\int T^\star_{ij} \neq 0$.

Poincaré noted that if some $\int T^\star_{ij}$ are nonzero then the system of sources is not in mechanical equilibrium until mechanical stresses $\int P^\star_{ij} = -\int T^\star_{ij}$ are developed to counter the electromagnetic stresses. The $P^\star_{ij}$ can be embedded in a 4-tensor $P_{\mu\nu}$ that includes the mechanical rest energy $m_{\rm mech}c^2 = \int P^\star_{00}$ and the mechanical momentum $c\mathbf{P}_{{\rm mech},i} = \int P^\star_{0i} = \int P^\star_{i0}$.

Then when one defines

$$P_\mu = \int (T_{0\mu} + P_{0\mu})\ d{\rm Vol}, \tag{8}$$

one has a true 4-vector, with

$$P_0 = U + m_{\rm mech}c^2, \tag{9}$$

$$P_i = c(\mathbf{P}_f + \mathbf{P}_{\rm mech})_i. \tag{10}$$

This formalism does not quite succeed in providing an independent interpretation of the "field momentum" $\mathbf{P}_f$ when sources are present. That is, only the sum $\mathbf{P}_f + \mathbf{P}_{\rm mech}$ has a dynamical meaning, where $\mathbf{P}_{\rm mech}$ includes a contribution associated with the mechanical stress that arise in response to electromagnetic forces.

4 $\mathbf{P}^\star_f$ Has No Dynamical Significance

There remains the specific topic of Question #26: what interpretation should be given when $\mathbf{P}^\star_f \neq 0$ in the "rest frame" of the sources? In view of the difficulty of giving any independent meaning to $\mathbf{P}_f$ when sources are present, this issue is secondary.

It is not very satisfactory to note that one can always find a frame in which $\mathbf{P}_f$ vanishes, since in general the center of mass of the sources will be moving in this frame.

Instead, we advocate a fairly trivial solution to the problem. Simply regard the value $\mathbf{P}^\star_f$ as a constant of the system without an interpretation of anything being in motion. This is a consistent view because the dynamical significance of momentum is in its derivative,

$$f_\mu = \frac{dP_\mu}{d\tau}, \tag{11}$$

where τ is the proper time, and in conservation laws, both of which are unaffected by an additive constant. In this sense, no dynamical meaning can be assigned to the value of $\mathbf{P}_f^\star$, and one can consistently choose not to give it any further interpretation.

We can amplify this point by recalling the Lorentz transformation of the 4-momentum $P_\mu = (U_f, c\mathbf{P}_f)$ in a boost by $\vec{\beta} = \mathbf{v}/c$ from the rest ($\star$) frame:

$$\mathbf{P}_f = \gamma \left(\mathbf{P}_f^\star + \frac{U_f^\star}{c^2} \mathbf{v} \right), \tag{12}$$

where $\gamma = 1/\sqrt{1 - (v/c)^2}$. Thus in a frame where the system moves with velocity $\mathbf{v}$, the part of the momentum that is proportional to velocity depends on the effective mass $U_f^\star/c^2$ in the rest frame, and not on the momentum $\mathbf{P}_f^\star$ in the rest frame. A nonzero value of $\mathbf{P}_f^\star$ in the rest frame has no dynamical effect on the momentum.

We have gotten used to electrons and photons having spin without being able to identify anything that rotates. So I propose that we not worry too much about a nonzero static value for the "field momentum" that has no dynamical consequence. Foregoing any interpretation of $\mathbf{P}_f^\star$ is even easier than for electron spin, since that latter has dynamical significance.

5 An Example

I append a further argument that shows how the "field momentum" $\mathbf{P}_f$ by itself does not consistently behave like a nonrelativistic momentum, whether or not its value in the rest frame of the sources is zero.

We consider a system that when at rest produces fields $\mathbf{E}_0$ and $\mathbf{B}_0$. The corresponding "field momentum"' $\mathbf{P}_0$ may or may not be zero, but in any case is a constant vector. O nly the time-dependent part of the "field momentum" will have relevance to $\mathbf{F} = d\mathbf{P}/dt$.

Next, consider the system when it is moving with center-of-mass velocity $\mathbf{v}$, where $v \ll c$. We suppose that there is no change in the state of the system relative to its center of mass, so fields $\mathbf{E}_0$ and $\mathbf{B}_0$ still hold in the rest frame of the system. Then the nonrelativistic limit of the transformation of the electromagnetic fields tells us that

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{v}}{c} \times \mathbf{B}_0, \tag{13}$$

$$\mathbf{B} = \mathbf{B}_0 + \frac{\mathbf{v}}{c} \times \mathbf{E}_0, \tag{14}$$

and so the "field momentum" associated with the moving system is

$$\mathbf{P}_f = \mathbf{P}_0 + \frac{1}{4\pi c^2} \int [(E_0^2 + B_0^2)\mathbf{v} + (\mathbf{E}_0 \cdot \mathbf{v})\mathbf{E}_0 + (\mathbf{B}_0 \cdot \mathbf{v})\mathbf{B}_0] d\mathrm{Vol}, \tag{15}$$

neglecting a term in $(v/c)^2$. The rate of change of this momentum is

$$\frac{d\mathbf{P}_f}{dt} = \frac{2U_0}{c^2}\mathbf{a} + \frac{1}{4\pi c^2} \int (\mathbf{E}_0 \cdot \mathbf{a})\mathbf{E}_0 + (\mathbf{B}_0 \cdot \mathbf{a})\mathbf{B}_0] d\mathrm{Vol}, \tag{16}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the acceleration of the system, and U_0 is the rest-frame field energy:

$$U_0 = \frac{1}{8\pi}\int (E_0^2 + B_0^2) d\mathrm{Vol}. \tag{17}$$

While, as expected, the constant value P_0 does not appear in the expression for the rate of change of "field momentum," this expression does not quite have the desired form, $m_{\rm eff}\mathbf{a}$. I infer that this is another demonstration of the view of Poincaré that the 'field momentum' $\mathbf{P}_f$ cannot be interpreted by itself when sources are present.

6 The Example of Question #26

Regarding the specific example [1] of nested electric and magnetic dipoles, it is easy to see that the diagonal elements of the electromagnetic stress tensor, T_{ii}, are nonvanishing. The sphere of charge and sphere of current-carrying coils would fly apart without some kind of glue. The resulting mechanical stresses change the rest mass of the system and, when it is in motion, its momentum by an amount comparable to the electromagnetic 'mass' and momentum contributions. Trying to interpret the electromagnetic momentum without considering the corresponding stress-induced changes in the mechanical momentum is counterproductive.

But the bottom line is that no meaningful dynamical interpretation can be given to the nonzero $\mathbf{P}_f^\star$ for that system in its rest frame.

7 Note added July 2012

Although we do not succeed in making the rest-frame field momentum $\mathbf{P}_f^\star$ part of a 4-vector, and hence part of special relativity, it continues to have a particular interest in that the total 3-momentum of a system at rest must be zero [6], so if $\mathbf{P}_f^\star$ is nonzero there must be a compensating "hidden" mechanical momentum,

$$\mathbf{P}_{\rm hidden}^\star = -\mathbf{P}_f^\star. \tag{18}$$

A substantial literature exists as to the character and significance of the "hidden" mechanical momentum (18).[1] In the author's view, "hidden" momentum for an unbounded (sub)system should be defined as,[2]

$$\mathbf{P}_{\rm hidden} = \mathbf{P} - \frac{U}{c^2}\mathbf{v}_{\rm center\ of\ energy}, \tag{19}$$

[1]In a classical context, where the magnetic field $\mathbf{B}$ is due to (Ampèrian) electric currents, nonzero "hidden" mechanical momentum is always associated with "moving parts" = the electrical currents [7], and one can, in some idealized models, relate the "hidden" mechanical momentum to differences in the relativistic momenta of moving charges in regions of different electric potential [6, 8, 9].

[2]A more general definition for bounded subsystems is given in [10].

where U and $\mathbf{P}$ are the total energy and momentum of the (sub)system and $\mathbf{v}_{\text{center of energy}}$ is the velocity of its center of mass/energy.[3,4]

References

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[3] There is no requirement in this definition that the center of mass/energy be at rest. If in the rest frame of a (sub)system its total momentum $\mathbf{P}^\star$ is nonzero, then this (sub)system has "hidden" momentum $\mathbf{P}^\star_{\text{hidden}} = \mathbf{P}^\star$ according to definition (19). In a frame where the total momentum is zero, the center of mass/energy (sub)system has nonzero velocity, so in this frame its "hidden" momentum is $\mathbf{P}_{\text{hidden}} = -(U/c^2)\mathbf{v}_{\text{center of energy}}$.

[4] in the example of Question #26 [1] (a shell of surface charge density that varies as $\cos\theta$ together with an electrically neutral shell with surface current that varies as $\sin\theta'$), the center of energy of its macroscopic electromagnetic field, and also of its "matter," are both at rest. The electromagnetic field momentum $\mathbf{P}^\star_f$ is nonzero if the axes that define angles θ and θ' are different, so this field momentum is a "hidden" momentum according to the definition (19). The "matter" subsystem then possesess nonzero momentum $\mathbf{P}^\star = -\mathbf{P}^\star_f$ such that the total momentum of the two subsystems is zero, and $\mathbf{P}^\star$ is also considered to be a "hidden" momentum.

Of course, the definition (19) does not explain the dynamical origin of the momentum of a (sub)system. In the example of Question #26 [1] the "hidden" momentum of the "matter" subsystem can be explained by the argument of [6] mentioned just before eq. (18). Another example of this type is discussed in [11], particularly secs. 2.7-8.

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