A Flapping Toy

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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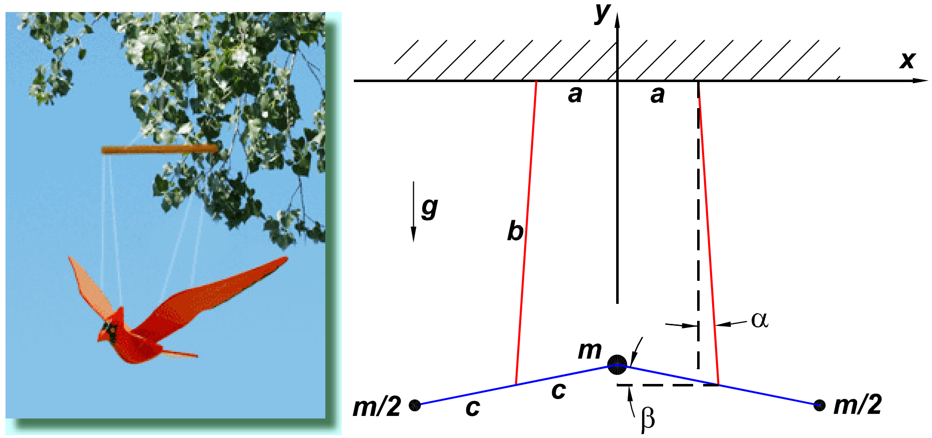
1 Problem

Deduce the frequency of small oscillations of the flapping toy shown in the figure below, supposing the central mass m moves only vertically, and the motion of the others masses is only in the x-y plane.



The flapping toy can be approximated as a mass m connected by two massless rods of length $2c$ to two masses $m/2$. The centers of the two rods are suspended from a horizontal plane by massless strings of length b, with distance $2a$ between the upper points of suspension. Lengths a and c similar, and both are smaller than b

2 Solution

This problem was suggested by Frank Moon.

A toy such as that modeled here can be purchased from several vendors [1, 2].

The static equilibrium configuration of the toy is such that its center of mass is such that the center of mass is as low as possible. Regarding half of the central mass m as being attached to the left rod, and the other half to the right arm, we see that the center of mass of each rod system is at the point of attachment to the string.

If $c \le a$ then there is only a single equilibrium state, with angles α and β, shown in the figure above, given by $\sin\alpha_0 = (a-c)/b \le 0$ and $\beta_0 = 0$. However, in this case angle α is always less than or equal to α_0 and does not oscillate about the equilibrium value; an analysis of small oscillations should be performed in terms of angle β.

If $c > a$ the strings are vertical at equilibrium. There are now two equilibrium states, with $\alpha_0 = 0$ and $\tan\beta_0 = \pm\sqrt{(c/a)^2 - 1}$. The frequency of small oscillation about either equilibrium state will be the same, but the amplitude of the oscillation of the central mass must by less than $\sqrt{c^2 - a^2}$ to be called "small"; otherwise the motion of the central mass will extend beyond both of its equilibrium points.

This system has only one degree of freedom if the motion is restricted to the x-y plane and the central mass moves vertically. In this case, the left and right rods make the same angle β to the horizontal, and the two strings make the same (small) angle α to the vertical. The system is equivalent to two identical systems consisting of a string, a rod, and two masses of $m/2$ at the ends of the rod, in which one mass is constrained to have $x = 0$.

To analyze the motion, it suffices to consider only one of the two subsystems, say the one at $x \geq 0$. The center of mass of the subsystem is at the point of attachment of the string to the rod. Thus, the potential energy of the subsystem is

$$V = mgy_{\rm cm} = -mgb\cos\alpha \approx -mgb + \frac{mgb}{2}\alpha^2. \tag{1}$$

This form suggests that it will be simpler to analyze the system in terms of coordinate α rather than β. However, as previously noted, small oscillations about equilibrium in angle α occur only when $c > a$.

2.1 Quick Analysis for $c > a$ using Angle α

In this case the potential (1) describes oscillations about $\alpha = 0$ with effective spring constant

$$k_{\rm eff} = mgb \qquad (c > a). \tag{2}$$

To relate α and β we use the constraint that the central mass is at $x = 0$,

$$0 = a + b\sin\alpha - c\cos\beta, \tag{3}$$

$$\cos\beta = \frac{a + b\sin\alpha}{c}, \tag{4}$$

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \frac{\sqrt{c^2 - (a + b\sin\alpha)^2}}{c} \approx \frac{\sqrt{c^2 - a^2}}{c}, \tag{5}$$

noting that $b\sin\alpha \ll c$ for very small α. To relate $\dot\beta$ to $\dot\alpha$ we return to eq. (4)

$$-\dot\beta\sin\beta = \frac{b\cos\alpha}{c}\,\dot\alpha, \tag{6}$$

$$\dot\beta = -\frac{b\cos\alpha}{\sqrt{c^2 - (a + b\sin\alpha)^2}}\,\dot\alpha \approx -\frac{b\dot\alpha}{\sqrt{c^2 - a^2}}. \tag{7}$$

The kinetic energy of the center of mass of the subsystem (which moves in a circle of radius b) is

$$T_{\rm cm} = \frac{mb^2}{2}\dot\alpha^2. \tag{8}$$

The kinetic energy of the rotational motion relative to the center of mass of the subsystem is

$$T_{\rm rel} = \frac{I\dot\beta^2}{2} = \frac{mc^2}{2}\dot\beta^2. \tag{9}$$

The total kinetic energy is, using eq. (7),

$$T = T_{\rm cm} + T_{\rm rel} \approx \frac{mb^2}{2}\frac{2c^2 - a^2}{c^2 - a^2}\dot\alpha^2 \qquad (c > a). \tag{10}$$

The effective mass of the subsystem in this case is

$$m_{\rm eff} = mb^2 \frac{2c^2 - a^2}{c^2 - a^2} \qquad (c > a). \tag{11}$$

The frequency of small oscillation is, recalling eq. (2),

$$\omega = \sqrt{\frac{k_{\rm eff}}{m_{\rm eff}}} = \sqrt{\frac{g(c^2 - a^2)}{b(2c^2 - a^2)}} \qquad (c > a), \tag{12}$$

which is small compared the frequencies $\sqrt{g/b}$ or $\sqrt{g/c}$ of simple pendula of the same characteristic lengths as components of the flapping toy.

For analyses of small oscillations when $c \le a$ we must use angle β rather than α as the coordinate. Following these analyses, we return to analyze the case $c > a$ using angle β.

2.2 Analysis for $c < a$ Using Angle β

To eliminate coordinate α in favor of β, we write eq. (3) as

$$\sin\alpha = \frac{c\cos\beta - a}{b}, \tag{13}$$

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \frac{\sqrt{b^2 - (c\cos\beta - a)^2}}{b} \approx \frac{\sqrt{b^2 - (a-c)^2}}{b} - \frac{c(a-c)}{2b\sqrt{b^2 - (a-c)^2}}\beta^2. \tag{14}$$

Then, the potential (1) can also be approximated as

$$V = -mgb\cos\alpha \approx -mg\sqrt{b^2 - (a-c)^2} + mg\frac{c(a-c)}{2\sqrt{b^2 - (a-c)^2}}\beta^2. \tag{15}$$

For $c < a$ the potential is springlike with effective spring constant

$$k_{\rm eff} = mg\frac{c(a-c)}{\sqrt{b^2 - (a-c)^2}} \qquad (c < a). \tag{16}$$

To express the kinetic energy in terms of $\dot\beta$ rather than $\dot\alpha$ we rewrite eq. (6) as

$$\dot\alpha = -\frac{c\dot\beta\sin\beta}{b\cos\alpha} \approx \frac{c\dot\beta\sin\beta_0}{b} = 0, \tag{17}$$

where the approximation holds for small oscillations about the equilibrium $\beta_0 = 0$ when $c \le a$. That is, only the rotational kinetic energy is significant when using coordinate β and $c \le a$.

$$T \approx T_{\rm rel} = \frac{mc^2}{2} \dot\beta^2, \qquad (18)$$

and the effective mass for coordinate β is

$$m_{\rm eff} = mc^2 \qquad (c < a). \qquad (19)$$

The frequency of small oscillation is, recalling eq. (16),

$$\omega = \sqrt{\frac{k_{\rm eff}}{m_{\rm eff}}} = \sqrt{\frac{g(a-c)}{bc\sqrt{b^2 - (a-c)^2}}} \qquad (c < a), \qquad (20)$$

Thus, the frequency of small oscillations goes to zero as c approaches a either from above or from below.

2.3 $c = a$

If lengths a and c and equal, then from eq. (14),

$$\cos\alpha \approx \sqrt{1 - \frac{c^2\beta^4}{4b^2}} \approx 1 - \frac{c^2\beta^4}{8b^2} \qquad (c = a), \qquad (21)$$

and the total (conserved) energy in terms of coordinate β is

$$E \approx T_{\rm rel} + V = \frac{mc^2\dot\beta^2}{2} + \frac{mgc^2\beta^4}{8b} = \frac{mgc^2\beta_{\rm max}^4}{8b} \qquad (c = a), \qquad (22)$$

where $\beta_{\rm max}$ is the maximum angle during an oscillation. Hence,

$$\dot\beta = \frac{d\beta}{dt} = \frac{1}{2}\sqrt{\frac{g}{b}}\sqrt{\beta_{\rm max}^4 - \beta^4} \qquad (c = a). \qquad (23)$$

During one quarter of an oscillation β varies between 0 and $\beta_{\rm max}$, so the period τ is given by

$$\begin{aligned}\tau &= 4\int_0^{\beta_{\rm max}} \sqrt{\frac{b}{g}} \frac{2}{\sqrt{\beta_{\rm max}^4 - \beta^4}}\, d\beta = \frac{8}{\beta_{\rm max}}\sqrt{\frac{b}{g}} \int_0^1 \frac{dz}{\sqrt{1 - z^4}} = \frac{1}{\beta_{\rm max}}\sqrt{\frac{2b}{\pi g}}\Gamma^2(1/4) \\ &= \frac{10.48}{\beta_{\rm max}}\sqrt{\frac{b}{g}} \quad (c = a), \qquad (24)\end{aligned}$$

using $z = \beta/\beta_{\rm max}$ and Gradshteyn and Ryzhik 3.166.16.

2.4 Analysis for $c > a$ Using Angle β

When analyzing the case that $c > a$ using coordinate β the approximation in eq. (14) is not appropriate (since the resulting potential (15) has no minimum). Instead, we make a Taylor expansion of $\cos\alpha$ as a function of β about its equilibrium value, given by $\cos\beta_0 = a/c$,

$$\cos\alpha = \frac{\sqrt{b^2 - (c\cos\beta - a)^2}}{b}, \qquad \cos\alpha(\beta_0) = 1, \tag{25}$$

$$\frac{d\cos\alpha}{d\beta} = \frac{c\sin\beta(c\cos\beta - a)}{b\sqrt{b^2 - (c\cos\beta - a)^2}}, \qquad \frac{d\cos\alpha(\beta_0)}{d\beta} = 0, \tag{26}$$

$$\begin{aligned} \frac{d^2\cos\alpha}{d\beta^2} &= \frac{c\cos\beta(c\cos\beta - a) - c^2\sin^2\beta}{b\sqrt{b^2 - (c\cos\beta - a)^2}} - \frac{c^2\sin^2\beta(c\cos\beta - a)^2}{b[b^2 - (c\cos\beta - a)^2]^{3/2}}, \\ \frac{d^2\cos\alpha(\beta_0)}{d\beta^2} &= -\frac{c^2\sin^2\beta_0}{b^2} = -\frac{c^2 - a^2}{b^2}, \end{aligned} \tag{27}$$

such that

$$\cos\alpha \approx 1 - \frac{c^2 - a^2}{2b^2}(\beta - \beta_0)^2, \tag{28}$$

and the potential can be approximated as

$$V = -mgb\cos\alpha \approx -mgb + mg\frac{c^2 - a^2}{2b}(\beta - \beta_0)^2 \qquad (c > a). \tag{29}$$

The effective spring constant for this analysis is

$$k_{\text{eff}} = mg\frac{c^2 - a^2}{b}. \tag{30}$$

When considering the kinetic energy in terms of coordinate β for $c > a$ we cannot neglect the contribution from the center of mass motion (as was possible for $c < a$). Recalling eq. (refe8a,

$$\dot\alpha = -\frac{c\dot\beta\sin\beta}{b\cos\alpha} \approx \frac{c\dot\beta\sin\beta_0}{b}, \tag{31}$$

so the kinetic energy can be approximated as

$$T = \frac{mb^2}{2}\dot\alpha^2 + \frac{mc^2}{2}(1 + \sin^2\beta_0)\dot\beta^2 = \frac{m(2c^2 - a^2)}{2}\dot\beta^2. \tag{32}$$

The effective mass for this analysis is

$$m_{\text{eff}} = m(2c^2 - a^2), \tag{33}$$

and the frequency of small oscillations is

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{g(c^2 - a^2)}{b(2c^2 - a^2)}} \qquad (c > a), \tag{34}$$

as found previously in sec. 2.1.

References

[1] http://www.flyingmobiles.com/

[2] Flying Toucan, http://www.woodentoys.com/