Radiation Damping of a Refrigerator Magnet

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(January 20, 2018; updated January 24, 2018)

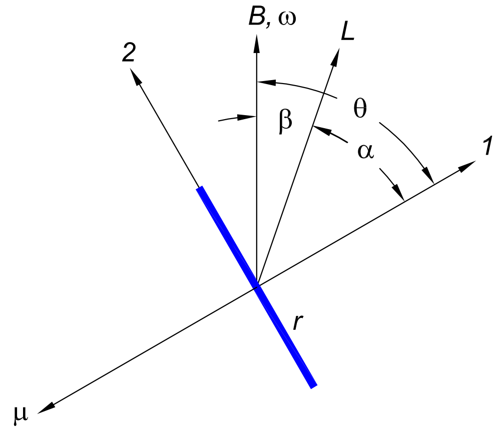
1 Problem

A refrigerator magnet in the form of a thin disc with magnetic moment $\boldsymbol{\mu}$ parallel to its axis is initially arranged for that axis to precess at angle θ around a uniform external magnetic field $\mathbf{B}$. What is the damping time of the motion due to magnetic dipole radiation?[1]

This problem was suggested by Shivaji Sondhi.

2 Solution

We label the unit vector along the axis of the magnet as $\hat{\mathbf{1}}$; the unit vector in the midplane of the magnet and in the $\hat{\mathbf{1}}$-$\mathbf{B}$ plane is $\hat{\mathbf{2}}$; then the unit vector $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ also lies in the midplane of the magnet. For the precession angular velocity $\boldsymbol{\omega}$ to be in the same direction as $\mathbf{B}$, the magnetic moment $\boldsymbol{\mu}$ is along the $-\hat{\mathbf{1}}$-axis (to be confirmed in eq. (5)), as in the figure below.



If the thin disc has mass m and radius r, then its inertia tensor has diagonal elements $(\mathsf{I}_{11}, \mathsf{I}_{22}, \mathsf{I}_{33}) = mr^2(2, 1, 1)/4$. The initial angular velocity $\boldsymbol{\omega} = \omega(\cos\theta, \sin\theta, 0)$ is parallel to $\mathbf{B}$, at angle θ to the $\hat{\mathbf{1}}$-axis. The "mechanical" angular momentum of the precessing disc is

$$\mathbf{L}_\omega = \mathsf{I} \cdot \boldsymbol{\omega} = \frac{mr^2\omega}{4}(2\cos\theta, \sin\theta, 0). \qquad (1)$$

In addition, there is angular momentum associated with the electrons whose magnetic moments comprise the total magnetic moment $\boldsymbol{\mu}$,

$$\mathbf{L}_\mu = -\frac{m_e c}{e}\boldsymbol{\mu} \approx -6 \times 10^{-8}\mu(1, 0, 0), \qquad (2)$$

[1] A related example concerns the spin down of magnetars [1].

in Gaussian units, where m_e and $e > 0$ are the mass and (magnitude of the) charge of an electron, and c is the speed of light in vacuum.

For the time being, we neglect the angular momentum $\mathbf{L}_{\rm EM}$ in the time-dependent electromagnetic field of the rotating magnetic dipole $\boldsymbol{\mu}$.

2.1 $L_\mu \ll L_\omega$

A neodymium magnet has mass density $\rho = m/{\rm Vol} = 7$ g/cm^3 and magnetization density $M = \mu/{\rm Vol} \approx 1000$ cgs units,[2] so $|L_\mu|/L_\omega \approx 6\times 10^{-8} M/\rho\, r^2\omega \approx 10^{-5}$ for a magnet with r and ω of order unity, and we will neglect $\mathbf{L}_m$ in the following.

In this approximation, the total initial angular momentum is

$$\mathbf{L} \approx \mathbf{L}_\omega = \left(\frac{mr^2\omega\cos\theta}{2}, \frac{mr^2\omega\sin\theta}{4}, 0\right) \equiv L(\cos\alpha, \sin\alpha, 0), \tag{3}$$

where the magnitude L of the angular momentum, and its angle α to the $\hat{\mathbf{1}}$-axis, are related by

$$L = \frac{mr^2\omega}{4}\sqrt{1+3\cos^2\theta}, \qquad \tan\alpha = \frac{\tan\theta}{2}, \qquad \cos\alpha = \frac{2\cos\theta}{\sqrt{1+3\cos^2\theta}}. \tag{4}$$

The (torque) equation of motion for steady precession is (neglecting $d\mathbf{L}_{\rm EM}/dt$),

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\tau} = \boldsymbol{\mu}\times\mathbf{B}, \qquad \omega L\sin\beta = \mu B\sin(\pi-\theta) = \mu B\sin\theta, \tag{5}$$

where the scalar equation is for the component along the $-\hat{\mathbf{3}}$-axis, $\beta = \theta - \alpha$, and $\boldsymbol{\omega}$ and $\mathbf{B}$ point in the same direction for $\boldsymbol{\mu}$ along the $-\hat{\mathbf{1}}$-axis. Then, recalling eq. (4),

$$\sin\beta = \sin\theta\cos\alpha - \cos\theta\sin\alpha = \sin\theta\cos\alpha(1-\cot\theta\tan\alpha) = \frac{\sin\theta\cos\alpha}{2} = \frac{\sin\theta\cos\theta}{\sqrt{1+3\cos^2\theta}}, \tag{6}$$

and the equation of motion (5) can be rewritten as

$$\frac{mr^2\omega^2}{4}\cos\theta = \mu B, \qquad \omega^2 = \frac{4\mu B}{mr^2\cos\theta}. \tag{7}$$

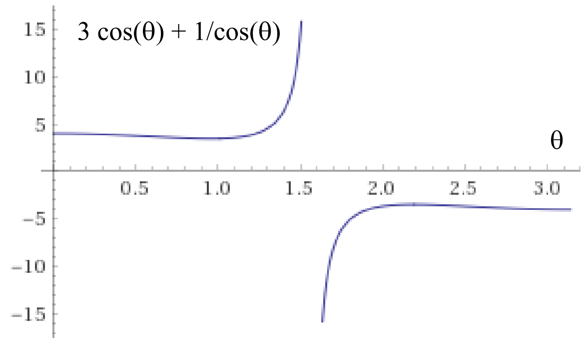
This implies that μB and $mr^2\omega$ are of the same order, and that ω is large compared to unity. The latter relation further suppresses the ratio $|L_m|/L_\omega$ considered previously, which reinforces that we can neglect the angular momentum $\mathbf{L}_m$. Equation (7) also indicates that steady precession could only exist for $\theta < \pi/2$.

However, when we consider the energy of the system, using eq. (7),

$$U = T + V = \frac{\boldsymbol{\omega}\cdot\mathsf{I}\cdot\boldsymbol{\omega}}{2} - \boldsymbol{\mu}\cdot\mathbf{B} = \frac{mr^2\omega^2}{8}(1+\cos^2\theta) + \mu B\cos\theta = \frac{\mu B}{2}\left(3\cos\theta + \frac{1}{\cos\theta}\right), \tag{8}$$

the kinetic and potential energies are comparable.

[2] This corresponds to residual induction (saturation remanence) of $B = 4\pi M \approx 13{,}000$ gauss.



Assuming that the motion is always just precession of the disc about $\mathbf{B}$, the time rate of change of the energy is,

$$\frac{dU}{dt} = \frac{\mu B \sin\theta}{2\cos^2\theta}\dot{\theta}(1 - 3\cos^2\theta). \tag{9}$$

The energy has a minimum at $\cos^2\theta = 1/3$, $\theta = 54.7°$.

At last, we come to consideration of the radiation of energy by the precessing magnetic moment,

$$\frac{dU}{dt} = -\frac{2\ddot{\mu}^2}{3c^3} = -\frac{2\mu^2\omega^4\sin^2\theta}{3c^3} = -\frac{32\mu^4 B^2\tan^2\theta}{3m^2r^4c^3}, \tag{10}$$

recalling, for example, eq. (71.5) of [2], and using our eq. (7). From eqs. (9) and (10) we have that

$$\dot{\theta} = -\frac{64\mu^3 B\sin\theta}{3m^2r^4c^3(1 - 3\cos^2\theta)}, \tag{11}$$

However, there is an inconsistency in the above analysis, in that the precessing disc always radiates energy, while the assumption of precession led to the identification of an energy minimum at $\theta = 54.7°$. It appears that the (adiabatic) model of instantaneous precession of the disc about $\mathbf{B}$ is not sufficiently accurate.

Another difficulty with this example is that the steady precession of a refrigerator magnet in, say, a magnetic field of 1 T = 10,000 gauss leads to internal stresses close to the breaking strength. As noted in [3] the maximum angular velocity or rotation of a ring of radius r and cross-sectional area A about its axis is related by

$$\omega_{\rm max}^2 r^2 = \frac{T_{\rm max}/A}{\rho}, \tag{12}$$

where $T_{\rm max}/A \approx 10^7$ dyne/cm^2 is the breaking strength. Comparing with eq. (7), we have that

$$\omega^2 r^2 = \frac{4\mu B}{m\cos\theta} = \frac{4MB}{\rho\cos\theta} = \frac{4\omega_{\rm max}^2 r^2}{\cos\theta} > \omega_{\rm max}^2 r^2, \tag{13}$$

for $M = 1000$, $B = 10,000$ gauss and $\rho = 7$ g/cm^3, so the precessing disc would break apart.

2.2 $L_\omega \ll L_\mu$

We now consider the case of very low angular velocity and weak external magnetic fields, in which we can neglect the angular momentum $\mathbf{L}_\omega$ compared to the tiny angular momentum $\mathbf{L}_\mu$ associated with the magnetic moment $\boldsymbol{\mu}$ of the refrigerator magnet. In this case, the angular momentum is along the $\hat{\mathbf{1}}$-axis,

$$\mathbf{L} \approx L_\mu(1,0,0), \tag{14}$$

and the equation of motion (5) reduces to

$$\omega = \frac{\mu B}{L_\mu} = \frac{eB}{m_e c}\,, \tag{15}$$

recalling eq. (2), such that the precession angular velocity is independent of angle θ.

The condition $L_\omega \ll L_\mu$ implies that

$$mr^2\omega \ll \frac{m_e c\mu}{e}\,, \qquad \frac{\rho \mathrm{Vol}\, r^2 eB}{m_e c} \ll \frac{m_e c M \mathrm{Vol}}{e}\,, \qquad B \ll \frac{m_e^2 c^2 M}{e^2 \rho} = 5 \times 10^{-15}\ \text{gauss}. \tag{16}$$

When this holds, the kinetic energy of rotation is also negligible compared to the potential energy, so the total energy is approximately

$$U \approx \mu B \cos\theta, \qquad \frac{dU}{dt} \approx -\mu B \dot{\theta} \sin\theta \tag{17}$$

The energy loss due to radiation is now, using eq. (14),

$$\frac{dU}{dt} = -\frac{2\ddot{\mu}^2}{3c^3} = -\frac{2\mu^2\omega^4 \sin^2\theta}{3c^3} = -\frac{2\mu^2 e^4 B^4 \sin^2\theta}{3m_e^4 c^7}\,, \tag{18}$$

which together with eq. (17) tells us that angle θ increases at the rate

$$\dot{\theta} = \frac{2\mu e^4 B^3 \sin\theta}{3m_e^4 c^7}\,, \tag{19}$$

until $\theta = \pi$ and the magnetic moment $\boldsymbol{\mu}$ becomes aligned with the magnetic field $\mathbf{B}$. This motion lasts for time

$$\Delta t \approx \frac{m_e^4 c^7}{\mu e^4 B^3} \approx 3 \times 10^{50}\ \text{s}, \tag{20}$$

for $\mu = M$ Vol $= 100$ cgs units and $B = 10^{-15}$ gauss. This is much longer than the age of the Universe.

2.3 General Equations of Motion for L

In this section we present a derivation of the equations of motion not presuming steady precession, and then make an estimate for the damping time associated with radiation of angular momentum (rather than radiation of energy) for the case $L_\mu \ll L_\omega$.

We work in the rotating frame of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$, and write the total angular velocity $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \omega_2\,\hat{\mathbf{2}} + \omega_3\,\hat{\mathbf{3}}. \tag{21}$$

The magnetic field $\mathbf{B}$ is constant in the lab frame, and always in the $\hat{\mathbf{1}}$-$\hat{\mathbf{2}}$ plane, at angle θ to the $\hat{\mathbf{1}}$-axis, which implies that

$$\omega_3 = -\dot{\theta}, \tag{22}$$

since the rotation due to positive ω_3 decreases θ. The time derivatives of the unit vectors are given by

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{1}} = \omega_3\,\hat{\mathbf{2}} - \omega_2\,\hat{\mathbf{3}}, = -\dot{\theta}\,\hat{\mathbf{2}} - \omega_2\,\hat{\mathbf{3}}, \tag{23}$$

$$\frac{d\hat{\mathbf{2}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{2}} = -\omega_3\,\hat{\mathbf{1}} + \omega_1\,\hat{\mathbf{3}} = \dot{\theta}\,\hat{\mathbf{1}} + \omega_1\,\hat{\mathbf{3}}, \tag{24}$$

$$\frac{d\hat{\mathbf{3}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{3}} = \omega_2\,\hat{\mathbf{1}} - \omega_1\,\hat{\mathbf{2}}, \tag{25}$$

The total angular momentum is

$$\mathbf{L} = \mathbf{L}_\omega + \mathbf{L}_\mu + \mathbf{L}_{\rm EM} = \frac{mr^2}{4}(2\omega_1\,\hat{\mathbf{1}} + \omega_2\,\hat{\mathbf{2}} + \omega_3\,\hat{\mathbf{3}}) + \frac{\mu m_e c}{e}\,\hat{\mathbf{1}} + \mathbf{L}_{\rm EM}. \tag{26}$$

The torque equation for the angular momentum is

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d\mathbf{L}_\omega}{dt} + \frac{d\mathbf{L}_\mu}{dt} + \frac{d\mathbf{L}_{\rm EM}}{dt} \\
&= \frac{mr^2}{4}\left(2\dot{\omega}_1\,\hat{\mathbf{1}} + \dot{\omega}_2\,\hat{\mathbf{2}} + \dot{\omega}_3\,\hat{\mathbf{3}} + 2\omega_1\,\frac{d\hat{\mathbf{1}}}{dt} + \omega_2\,\frac{d\hat{\mathbf{2}}}{dt} + \omega_3\,\frac{d\hat{\mathbf{3}}}{dt}\right) + \frac{\mu m_e c}{e}\,\frac{d\hat{\mathbf{1}}}{dt} + \frac{d\mathbf{L}_{\rm EM}}{dt} \\
&= \frac{mr^2}{4}\left(2\dot{\omega}_1\,\hat{\mathbf{1}} + (\dot{\omega}_2 - \omega_1\dot{\theta})\,\hat{\mathbf{2}} - (\ddot{\theta} + \omega_1\omega_2)\,\hat{\mathbf{3}}\right) + \frac{\mu m_e c}{e}(\dot{\theta}\,\hat{\mathbf{2}} - \omega_2\,\hat{\mathbf{3}}) + \frac{d\mathbf{L}_{\rm EM}}{dt} \\
&= \boldsymbol{\mu}\times\mathbf{B} = -\mu B\sin\theta\,\hat{\mathbf{3}}.
\end{aligned} \tag{27}$$

We could cast this as an equation of motion for the “mechanical” angular momentum by the rearrangement,

$$\begin{aligned}
\frac{d\mathbf{L}_{\boldsymbol{\omega}}}{dt} &= \frac{mr^2}{4}\left(2\dot{\omega}_1\,\hat{\mathbf{1}} + (\dot{\omega}_2 - \omega_1\dot{\theta})\,\hat{\mathbf{2}} - (\ddot{\theta} + \omega_1\omega_2)\,\hat{\mathbf{3}}\right) \\
&= -\mu B\sin\theta\,\hat{\mathbf{3}} - \frac{\mu m_e c}{e}(\dot{\theta}\,\hat{\mathbf{2}} - \omega_2\,\hat{\mathbf{3}}) - \frac{d\mathbf{L}_{\rm EM}}{dt}\,.
\end{aligned} \tag{28}$$

The last term of eq. (28) could be regarded as the reaction to the radiation of angular momentum.

If, as in sec. 2.1, we neglect the last two terms of eq. (28), then eq. (28) tells us that

$$\dot{\omega}_1 = 0, \qquad \dot{\omega}_2 - \omega_1\dot{\theta} = 0, \qquad \omega_1\omega_2 + \ddot{\theta} = \frac{4\mu B\sin\theta}{mr^2} \qquad (L_\omega \gg L_\mu,\ L_{\rm EM}). \tag{29}$$

For steady precession, $\dot{\theta} = -\omega_3 = 0$, so ω_1 and ω_2 are constant according to the first and second of eq. (29). The total angular velocity $\boldsymbol{\omega}$ lies in the $\hat{\mathbf{1}}$-$\hat{\mathbf{2}}$ plane, and is parallel to $\mathbf{B}$ for steady precession. Then, since $\mathbf{B}$ makes angle θ to the $\hat{\mathbf{1}}$-axis by definition, $\omega_1 = \omega\cos\theta$, $\omega_2 = \omega\sin\theta$, and the third of eq. (29) becomes

$$\omega^2 = \frac{4\mu B}{mr^2\cos\theta} \qquad (L_\omega \gg L_\mu,\ L_{\rm EM}), \tag{30}$$

as previously found in eq. (7).

While steady precession cannot occur, in general, with the inclusion the last term of eq. (28), we might estimate $\dot{\theta}$ from the $\hat{\mathbf{2}}$-component of this equation (still neglecting $\mathbf{L}_\mu$) as

$$\frac{mr^2\omega_1}{4}\,\dot{\theta} = \frac{mr^2\omega\sin\theta}{4}\,\dot{\theta} \approx \left.\frac{d\mathbf{L}_{\rm EM}}{dt}\right|_2 = \frac{2}{3c^3}[\dot{\boldsymbol{\mu}}\times\ddot{\boldsymbol{\mu}}]_2 = \frac{dU}{dt}\frac{\omega_2}{\omega^2} \approx \frac{2\mu^2\omega^3\cos\theta\sin^2\theta}{3c^3}\,, \tag{31}$$

using eq. (75.7) of [2].[3] Then, supposing the motion to be approximately steady precession, we use eq. (30) in eq. (31), and ignore the sines and cosines, to estimate that the (nonexponential) radiation damping through angle $\Delta\theta \approx 1$ takes time Δt related by

$$\dot{\theta} \approx \frac{\Delta\theta}{\Delta t} \approx -\frac{32\mu^4B^2}{3m^2r^4c^3}\,, \qquad \Delta t \approx \frac{3m^2r^4c^3}{32\mu^4B^2} = \frac{3\rho^2r^4c^3}{32M^4{\rm Vol}^2B^2} \approx 3\times10^{15}\ {\rm s} \approx 10^8\ {\rm yr}, \tag{32}$$

for $\rho = 7$ g/cm^3, $r = 1$ cm, Vol $= 0.1$ cm^3, magnetization density $M = 10^3$ cgs units and $B = 3000$ gauss (*i.e.*, for a field such that the magnet is not stressed beyond its breaking strength by its rotation).

This estimate of the damping time (for $L_\mu \ll L_\omega$) scales as $1/\mu^4B^2$, in contrast to the estimate (20) which scales as $1/\mu B^3$ (for $L_\omega \ll L_\mu$, *i.e.*, for very weak fields).[4]

References

[1] K.T. McDonald, *Magnetars* (Nov. 29, 1998),
http://physics.princeton.edu/~mcdonald/examples/magnetar.pdf

[2] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, 1975), http://physics.princeton.edu/~mcdonald/examples/EM/landau_ctf_71.pdf

[3] K.T. McDonald, *Rayleigh's Spinning Ring* (July 12, 2017),
http://physics.princeton.edu/~mcdonald/examples/ring.pdf

[4] K.T. McDonald, *Electrodynamics Problem Set 8* (2001),
http://physics.princeton.edu/~mcdonald/examples/ph501set8.pdf

[3]See also prob. 5 of [4].

[4]For $L_\omega \ll L_\mu$, eq. (27) can be interpreted as describing steady precession as in sec. 2.1 plus damping due to radiation of angular momentum, with the latter related by

$$\frac{\mu m_e c}{e}\dot{\theta} = \left.\frac{d\mathbf{L}_{\rm EM}}{dt}\right|_2, \qquad \dot{\theta} = \frac{2e\mu B^3\sin^3\theta}{3m_e^4c^7}\,, \tag{33}$$

using eqs. (15) and (31). This has essentially the same form as eq. (19), so an estimate of the damping time Δt based on radiation of angular momentum is the same as that of eq. (20) based on radiation of energy.