Slepian's Faster-Than-Light Wave

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

In 1949, Slepian [1] posed an academic puzzler about a plane electromagnetic wave with vector potential,

$$\mathbf{A} = A\cos[k(x - vt)]\cos(k\,\Gamma\,y)\,\hat{\mathbf{z}}, \qquad \text{where} \qquad \Gamma = \sqrt{\frac{v^2}{c^2} - 1}, \tag{1}$$

and c is the speed of light in vacuum. This waveform obeys $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, as well as the free-space wave equation

$$\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\,, \tag{2}$$

although the waveform (1) ostensibly propagates at speed v rather than c.[1] Hence, supposing that the scalar potential V is zero,[2] the electromagnetic fields,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{A}}{\partial t}\,, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{3}$$

are solutions to Maxwell's equations, and propagate in the x-direction with velocity v, which can be greater than c.

Is this possible? Could such waves be used to send signals faster than light?[3]

2 Solution

The vector potential (1) can be written as the real part of the form,

$$\begin{aligned}\mathbf{A} &= \frac{A}{2}\,e^{i(kx-\omega t)}\big(e^{ik\Gamma y} + e^{-ik\Gamma y}\big)\,\hat{\mathbf{z}} = \frac{A}{2}\,e^{i(kx+k\Gamma y-\omega t)}\,\hat{\mathbf{z}} + \frac{A}{2}\,e^{i(kx-k\Gamma y-\omega t)}\,\hat{\mathbf{z}} \\ &\equiv \frac{A}{2}\,e^{i(\mathbf{k}_+\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{z}} + \frac{A}{2}\,e^{i(\mathbf{k}_-\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{z}} \equiv \mathbf{A}_+ + \mathbf{A}_-, \end{aligned} \tag{4}$$

[1] Slepian also noted that if $v < c$, then $\cos(k\Gamma y) = \cosh(k\Gamma' y)$ where $\Gamma' = i\Gamma = \sqrt{1 - v^2/c^2}$, and the waveform (1) again propagates with speed v.

[2] The vector potential (1), together with scalar potential $V = 0$, satisfy the conditions for the Coulomb, Gibbs, Lorenz and velocity gauges. See, for example, [2].

[3] This problem is distinct from comments in [3, 4] that a Helmholtz decomposition of the electric field into so-called irrotational and rotational fields (aka longitudinal/parallel and solenoidal/transverse fields); see, for example, [5]), $\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp$, leads to a wave equation for $\mathbf{E}_\perp$, but not for $\mathbf{E}_\parallel$. This author concurs with discussion in [6] that the Helmholtz decomposition is a formal construct requiring knowledge of the fields throughout all of space, so this decomposition cannot be made on the basis of local measurement. That is only the (locally measurable) total electric field has physical significance; it is not possible to use the component $\mathbf{E}_\parallel$ to send signals faster than light.

where $\omega = vk$ and $\mathbf{k}_\pm = k\,\hat{\mathbf{x}} \pm k\Gamma\,\hat{\mathbf{y}}$.[4]

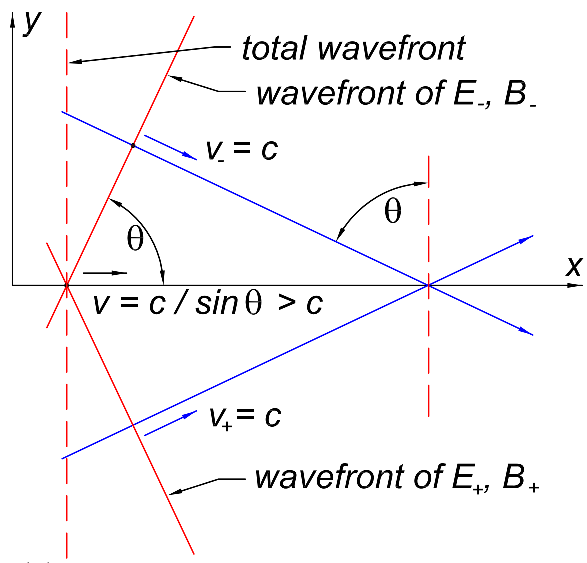
That is, Slepian's wave (1) is the sum of two plane waves, with wave vectors $\mathbf{k}_\pm$, each of which propagates with phase velocity $v_{p\pm}$ equal to c,[5]

$$v_{p\pm} = \frac{\omega}{|\mathbf{k}_\pm|} = \frac{\omega}{k\sqrt{1+\Gamma^2}} = \frac{vk}{k(v/c)} = c, \tag{6}$$

but with different directions $\hat{\mathbf{k}}_\pm$ given by,

$$\hat{\mathbf{k}}_\pm = \frac{\mathbf{k}_\pm}{|\mathbf{k}_\pm|} = \frac{k\,\hat{\mathbf{x}} \pm k\Gamma\,\hat{\mathbf{y}}}{k\sqrt{1+\Gamma^2}} = \frac{c}{v}(\hat{\mathbf{x}} \pm \Gamma\,\hat{\mathbf{y}}), \qquad \text{and} \qquad \mathbf{v}_{p\pm} = v_{p\pm}\,\hat{\mathbf{k}}_\pm = c\,\hat{\mathbf{k}}_\pm = \frac{c^2}{v}(\hat{\mathbf{x}} \pm \Gamma\,\hat{\mathbf{y}}). \tag{7}$$

The speed of the sum of two electromagnetic plane waves, each of which propagates with speed c, can have speed greater than c. This is sometimes called the "scissor effect", that the point of intersection of the blades of a pair of scissors is larger than the velocity of the blades themselves.



$$\sin\theta = \frac{k_{\pm,x}}{|\mathbf{k}_\pm|} = \frac{c}{v}\,.$$

So, Slepian's form (1) is a valid electromagnetic wave, and in fact corresponds to the so-called TE_{10} mode of a rectangular waveguide, as discussed in sec. 2.1 below.

However, if such a wave were to be used to send a signal, say from the origin at time $t = 0$, via some appropriate modulation of the waveform, that modulation must be applied

[4]For completeness, the electric field is $\mathbf{E} = i\omega\mathbf{A} \equiv \mathbf{E}_+ + \mathbf{E}_-$ where $\mathbf{E}_\pm = i\omega\mathbf{A}_\pm$, and the magnetic field is, using eq. (6),

$$\begin{aligned}\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} = ik(\Gamma\,\hat{\mathbf{x}} - \hat{\mathbf{y}})\frac{A}{2}\,e^{i(\mathbf{k}_+\cdot\mathbf{r}-\omega t)} - ik(\Gamma\,\hat{\mathbf{x}} + \hat{\mathbf{y}})\frac{A}{2}\,e^{i(\mathbf{k}_-\cdot\mathbf{r}-\omega t)} \equiv \mathbf{B}_+ + \mathbf{B}_- \\ &= \frac{\mathbf{k}_+}{|\mathbf{k}_+|}\frac{|\mathbf{k}_+|}{\omega}\times i\omega\frac{A}{2}\,e^{i(\mathbf{k}_+\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{z}} + \frac{\mathbf{k}_-}{|\mathbf{k}_-|}\frac{|\mathbf{k}_-|}{\omega}\times i\omega\frac{A}{2}\,e^{i(\mathbf{k}_-\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{z}} = \hat{\mathbf{k}}_+\times\frac{\mathbf{E}_+}{c} + \hat{\mathbf{k}}_-\times\frac{\mathbf{E}_-}{c}\,.\end{aligned} \tag{5}$$

[5]If $v < c$, we can write the vector potential (1) as $\mathbf{A} = A\cos[k(x - vt)]\cosh(k\Gamma'\,y)\,\hat{\mathbf{z}}$, where $\Gamma' = \sqrt{1 - v^2/c^2}$. This waveform diverges as $y \to \pm\infty$, so is of less physical interest than eq (1) with $v > c$.

For $v = 0$, the vector potential becomes the static form $\mathbf{A} = A\cos(kx)\cosh(ky)\,\hat{\mathbf{z}}$, for which the electric field is zero and the magnetic field is $\mathbf{B} = kA\cosh(ky)[\cos(kx)\,\hat{\mathbf{x}} + \sin(kx)\,\hat{\mathbf{y}}]$, which is formally valid but unphysical.

over the entire wavefront at $t = 0$. But, since the wavefront is infinite in extent, the message must be sent out from the origin at a much earlier time (strictly, at $t = -\infty$), to prepare the wavefront over its entire extent to be able to send the desired message at $t = 0$. While this message reaches an observer at distance d from the origin at time $t = d/v < d/c$, the time elapsed since the message was sent out from the origin to be encoded on the wavefront at $t = 0$ is much larger than d/c (and strictly infinite).

In sum, signals cannot be sent faster than light by a wave that exhibits the "scissor effect", even though the nominal phase velocity of the wave exceeds the speed of light.

Slepian's example is related to so-called Bessel beams (aka diffraction-free beams) that received some notoriety in the late 1980's [7, 8, 9, 10].

2.1 TE_{10} Mode of a Rectangular Waveguide

Electromagnetic waves inside cylinders with perfectly conducting walls were first discussed in 1888 by Heaviside, pp. 443-467 of [11], in 1893 by J.J. Thomson in sec. 300 of [12], and then by Rayleigh in 1897 [13]. Technological development of such waveguides was precipitated by two important papers from Bell Labs in 1936 [14, 15].[6] The realization that it would be more practical to use rectangular, rather than circular, guides may be due to Brillouin [16].

The electromagnetic fields of the TE_{10} mode with angular frequency ω in a rectangular guide whose interior is $-a/2 < y < a/2$ and $-b/2 < z < b/2$ can be deduced from a sinusoidal form for $\mathbf{E} = E_z(x, z, t)\,\hat{\mathbf{z}}$ that obeys the perfect-conductor boundary condition $E_z(y = -a/2) = E_z(y = a/2) = 0$. Then, the fields are given by the real parts of (see, for example, [17])

$$
\begin{aligned}
E_z &= E_0 \cos\frac{\pi y}{a}\, e^{i(kx-\omega t)}, && (8)\\
B_y = \frac{i}{\omega}\frac{\partial E_z}{\partial x} &= -\frac{k_g}{\omega} E_0 \sin\frac{\pi y}{a}\, e^{i(kx-\omega t)}, && (9)\\
B_x = -\frac{i}{\omega}\frac{\partial E_z}{\partial y} &= -\frac{i\pi}{\omega a} E_0 \cos\frac{\pi y}{a}\, e^{i(kx-\omega t)}, && (10)
\end{aligned}
$$

in SI units, and the guide wave number k (often called k_g since it does not equal ω/c) is given by

$$
k = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}}, \tag{11}
$$

where c is the speed of light in vacuum, such that eqs. (8)-(10) satisfy the wave equation $\nabla^2\psi = \partial^2\psi/\partial(ct)^2$.

The TE_{10} fields (8)-(10) can also be thought of as the sum of a pair of free-space waves that zig-zag down the guide at angle θ as shown on the previous page, and in the figure below from [16].[7]

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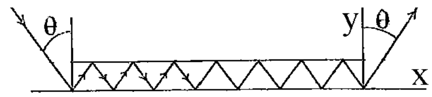
[6] The latter paper also discussed what would now be called optical fibers.

[7] The zig-zag character for guided waves was also discussed in Chap. 24, Vol. II of [18], but the angle θ used there is the complement of the angle used here and in [16, 17].



$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{E_0}{2}\left(e^{i(\mathbf{k}_+\cdot\mathbf{r}-\omega t)} + e^{i(\mathbf{k}_-\cdot\mathbf{r}-\omega t)}\right)\hat{\mathbf{z}} = E_0\cos\left(\frac{\omega}{c}\cos\theta\, y\right) e^{i(kx-\omega t)}\,\hat{\mathbf{z}}, \qquad (12)$$

$$\mathbf{B} = \frac{\mathbf{k}_+\times\mathbf{E}_+}{\omega} + \frac{\mathbf{k}_-\times\mathbf{E}_-}{\omega}, \qquad \mathbf{k}_\pm = \frac{\omega}{c}\sin\theta\,\hat{\mathbf{x}} \pm \frac{\omega}{c}\cos\theta\,\hat{\mathbf{y}} = k\,\hat{\mathbf{x}} \pm k\Gamma\,\hat{\mathbf{y}}. \qquad (13)$$

The requirement that $E_z(y = \pm a/2) = 0$ implies that $(\omega/c)\cos\theta = \pi/a$, so that $k = (\omega/c)\sin\theta$ is given by eq. (11). Since also $k = \omega/v$, we have that $\sin\theta = kc/\omega = c/v$, and hence $\Gamma = (\omega/kc)\cos\theta = (v/c)\sqrt{1 - c^2/v^2} = \sqrt{v^2/c^2 - 1}$, in agreement with eqs. (1) and (4).

2.2 Phase and Group Velocities

If we use a waveguide of finite cross section $a \times b$, as in sec. 2.1 above, to implement Slepian's wave (1), we avoid the issue of infinite time required to modulate the wave. Then, perhaps we could used the waveguide to send signals faster than light.

However, if the wave (8)-(10) is to send a signal via modulation of that wave, the resulting wave cannot simply have a single (angular) frequency ω, but must be the superposition of a set of waves of different frequencies. Then, we should speak of a wave packet, and not just the monochromatic wave (4) whose phase velocity is $v_p = \omega/k = v > c$. The velocity of the message is better described by the group velocity $v_g = d\omega/dk$.[8] From eq. (11), we have that $\omega = \sqrt{k^2c^2 + \pi^2/a^2}$, so the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{k^2c^2 + \frac{\pi^2}{a^2}} = \frac{kc^2}{\omega} = \frac{c^2}{v} < c, \qquad (14)$$

and also $v_p v_g = c^2$.

Thus, waveguides cannot send signals faster than light, even though the guide phase velocity $v_p = \omega/k$ does exceed the speed of light.

2.3 Comments

The commentary by Peterson[9] on Slepian's puzzler [1] attempted to raise the issues of phase and group velocity, but was not successful in convincing Slepian of their relevance. In his reply to Peterson of Jan. 1951, Slepian briefly mentioned that his waves occur in waveguides.

While waveguides had been studied theoretically since Heaviside's initial effort [11] in 1888, they were little studied in the lab until 1936, when it was noted, both theoretically

[8]The concept of group velocity appears to have been first enunciated by Hamilton in 1839 in lectures of which only abstracts were published [19]. The first recorded observation of the group velocity of a (water) wave is due to Russell in 1844 [20]. However, widespread awareness of group velocity dates from 1876 when Stokes posed it as the topic of a Smith's Prize examination paper [21]. The early history of group velocity has been reviewed by Havelock [22].

[9]See the Letter to the Editor on p. 93, appended to [1].

and experimentally (see Fig. 6 of [14]), that the phase velocity of the simplest guided wave exceeds the speed of light. Apparently this was something of a surprise, although this phenomenon had been implicit in the theory for 50 years.

It had been noted around 1900 that propagation of light through gaseous media at frequencies very close to that of absorption "lines" is associated with "anomalous dispersion", in which case the phase velocity can exceed that of light (although the attenuation of the light is very severe in this regime) [23, 24, 25, 26]. Such phenomena seemed to be in conflict with the notion of the speed of light as a kind of "limiting" velocity, and led to an understanding, due to Sommerfeld and Brillouin (1914) [27, 28, 29], that the phase velocity is not necessarily the "signal velocity", which latter is always less than or equal to c.

For wave propagation in "vacuum", as supposed in the present example, the "signal velocity" is the group velocity (for a signal based an a narrow enough range of frequencies). This was pointed out by Brillouin in sec. IV of his 1936 paper [16], which included eq. (14), along with the decomposition of the wave (4) whose phase velocity exceeds c into two zig-zag waves each with phase velocity c.

References

[1] J. Slepian, *Waves of All Velocities*, Elec. Eng. **68**, 1080 (1949), http://physics.princeton.edu/~mcdonald/examples/EM/slepian_ee_68_1080_49.pdf

[2] K.-H. Yang and K.T. McDonald, *Formal Expressions for the Electromagnetic Potentials in Any Gauge* (Feb. 25, 2015), http://physics.princeton.edu/~mcdonald/examples/gauge.pdf

[3] C.W. Wong, *Maxwell equations and the redundant gauge degree of freedom*, Eur. J. Phys. **30**, 1401 (2009), http://physics.princeton.edu/~mcdonald/examples/EM/wong_ejp_30_1401_09.pdf

[4] C.W. Wong, *Spurious causality hiding an action at a distance* (Dec. 1, 2010), https://arxiv.org/abs/1011.4541

[5] K.T. McDonald, *The Helmholtz Decomposition and the Coulomb Gauge* (April 17, 2008), http://physics.princeton.edu/~mcdonald/examples/helmholtz.pdf

[6] J.D. Jackson, *Comment on 'Maxwell equations and the redundant gauge degree of freedom'*, Eur. J. Phys. **31**, L79 (2010), http://physics.princeton.edu/~mcdonald/examples/EM/jackson_ejp_31_L79_10.pdf

[7] J. Durnin, *Exact solutions for nondiffracting beams. I. The scalar theory*, J. Opt. Soc. Am. A **4**, 651 (1987), http://physics.princeton.edu/~mcdonald/examples/optics/durnin_josa_a4_651_87.pdf

[8] J. Durnin, J.J. Miceli, Jr and J.H. Eberly, *Diffraction-free beams*, Phys. Rev. Lett. **58**, 1499 (1987), http://physics.princeton.edu/~mcdonald/examples/optics/durnin_prl_58_1499_87.pdf

[9] D. Mugnai, A. Ranfagni and R. Ruggeri, *Observation of Superluminal Behavior in Wave Propagation*, Phys. Rev. Lett. **84**, 4830 (2000), http://physics.princeton.edu/~mcdonald/examples/optics/mugnai_prl_84_4830_00.pdf

[10] K.T. McDonald, *Bessel Beams* (June 17, 2000),
http://physics.princeton.edu/~mcdonald/examples/bessel.pdf

[11] O. Heaviside, *Electrical Papers*, Vol. 2 (Macmillan, 1894),
http://physics.princeton.edu/~mcdonald/examples/EM/heaviside_electrical_papers_2.pdf

[12] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893),
http://physics.princeton.edu/~mcdonald/examples/EM/thomson_recent_researches_in_electricity.pdf

[13] Lord Rayleigh, *On the Passage of Electric Waves through Tubes, or the Vibrations of Dielectric Cylinders*, Phil. Mag. **43**, 125 (1897),
http://physics.princeton.edu/~mcdonald/examples/EM/rayleigh_pm_43_125_97.pdf

[14] G.C. Southworth, *Hyper-Frequency Wave Guides—General Considerations and Experimental Results*, Bell Syst. Tech. J. **15**, 284 (1936),
http://physics.princeton.edu/~mcdonald/examples/EM/southworth_bstj_15_284_36.pdf

[15] J.R. Carson, S.P. Mead, and S.A. Schelkunoff, *Hyper-Frequency Wave Guides—Mathematical Theory*, Bell Syst. Tech. J. **15**, 310 (1936),
http://physics.princeton.edu/~mcdonald/examples/EM/carson_bstj_15_310_36.pdf

[16] L. Brillouin, *Propagation d'ondes électromagnétique dans un tayau*, Rev. Gén. Élec. **40**, 227 (1936), http://physics.princeton.edu/~mcdonald/examples/EM/brillouin_rge_40_227_36.pdf
Much of this article is translated in Chap. VI of [29].

[17] S.J. Orfanidis, *Waveguides*, http://www.ece.rutgers.edu/~orfanidi/ewa/ch09.pdf

[18] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. II (Addison-Wesley, 1964), http://www.feynmanlectures.caltech.edu/II_24.html#Ch24-S8

[19] W.R. Hamilton, *Researches respecting vibration, connected with the theory of light*, Proc. Roy. Irish Acad. **1**, 267, 341 (1839),
http://physics.princeton.edu/~mcdonald/examples/optics/hamilton_pria_1_267_41.pdf
http://physics.princeton.edu/~mcdonald/examples/optics/hamilton_pria_1_341_41.pdf

[20] J.S. Russell, *Report on Waves*, Brit. Assoc. Reports (1844), pp. 311-390,
http://physics.princeton.edu/~mcdonald/examples/fluids/russell_bar_311_44.pdf
This report features the first recorded observations of solitary waves (p. 321) and of group velocity (p. 369).

[21] G.G. Stokes, Problem 11 of the Smith's Prize examination papers (Feb. 2, 1876), in *Mathematical and Physical Papers*, Vol. 5 (Johnson Reprint Co., New York, 1966), p. 362,
http://physics.princeton.edu/~mcdonald/examples/fluids/stokes_mathematical_and_physical_papers_v5.pdf

[22] T.H. Havelock, *The Propagation of Disturbances in Dispersive Media* (Cambridge U. Press, 1914), http://physics.princeton.edu/~mcdonald/examples/EM/havelock_waves_14.pdf

[23] W. Voigt, *Ueber die Aenderung der Schwingungsform des Lichtes beim Fortschreiten in einem dispergirenden oder absorbirenden Mittel*, Ann. d. Phys. **68**, 589 (1899),
http://physics.princeton.edu/~mcdonald/examples/EM/voigt_ap_68_598_99.pdf
Weiteres zur die Aenderung der Schwingungsform des Lichtes beim Fortschreiten in einem dispergirenden und absorbirenden Mittel, Ann. d. Phys. **4**, 209 (1901),
http://physics.princeton.edu/~mcdonald/examples/EM/voigt_ap_4_209_01.pdf

[24] A. Schuster, *The Propagation of Waves through dispesive Media*, in *Festschrift Ludwig Boltzmann* (Barth, 1904), p. 569,
http://physics.princeton.edu/~mcdonald/examples/EM/schuster_flb_569_04.pdf

[25] A. Sommerfeld, *Ein Einwand gegen die Relativtheorie der Elektrodynamik und seine Beseitigung*, Phys. Z. **8**, 841 (1907),
http://physics.princeton.edu/~mcdonald/examples/GR/sommerfeld_pz_8_841_07.pdf
Translation: *An Objection against the Theory of Relativity and its Removal*,
http://physics.princeton.edu/~mcdonald/examples/GR/sommerfeld_pz_8_841_07_english.pdf

[26] P. Ehrenfest, *Mißt der Aberratiomswinkel im Fall einer Dispersion des Äthers die Wellengeschwindigkeit?* Ann. d. Phys. **33**, 1571 (1910),
http://physics.princeton.edu/~mcdonald/examples/EM/ehrenfest_ap_33_1571_10.pdf

[27] A. Sommerfeld, *Über die Fortpflanzung des Lichtes in dispergierenden Medien*, Ann. d. Phys. **14**, 177 (1914), http://physics.princeton.edu/~mcdonald/examples/EM/sommerfeld_ap_14_177_14.pdf
This article is translated in Chap. II of [29].

[28] L. Brillouin, *Über die Fortpflanzung des Lichtes in dispergierenden Medien*, Ann. d. Phys. **14**, 203 (1914), http://physics.princeton.edu/~mcdonald/examples/EM/brillouin_ap_14_203_14.pdf
This article is translated in Chap. III of [29].

[29] L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, 1960),
http://physics.princeton.edu/~mcdonald/examples/EM/brillouin_waves_60.pdf