

A Relativistic Electron Can't Extract Net Energy from a Long Laser Pulse

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In this note I write out the argument of Sprangle *et al.* [1] at some length – and add to it a result, eq. (9), that I have not seen published elsewhere. The specific point of this argument is that the longitudinal component of the laser field near a focus, neglected in the analysis of [2], cancels any energy transferred to a relativistic electron by the transverse component.¹

The Gaussian approximation to a laser beam that propagates along the $+z$ axis and is polarized in the \mathbf{x} direction is, to leading order:

$$\mathbf{E} = \frac{E_0 \hat{\mathbf{x}}}{\sqrt{1 + \varsigma^2}} e^{-i \tan^{-1} \varsigma} e^{i \rho^2 \varsigma / (1 + \varsigma^2)} e^{-\rho^2 / (1 + \varsigma^2)} e^{i \varphi}, \quad (1)$$

where $\rho^2 = (x^2 + y^2)/w_0^2$, the radius of the waist is w_0 , $\varsigma = z/z_0$, the Rayleigh range is

$$z_0 = \frac{\pi w_0^2}{\lambda} = \frac{k w_0^2}{2}, \quad (2)$$

the phase is

$$\varphi = kz - \omega t, \quad (3)$$

the frequency of the wave is ω , the wave number is $k = \omega/c$, the wavelength is λ and c is the speed of light.

Expression (1) describes the main features of the focus of the laser beam, but it does not satisfy the Maxwell equation $\nabla \cdot \mathbf{E} = 0$. To see this, note that a divergence-free field that points only in the x direction cannot vary with x .

We find below that an electric field which satisfies Maxwell's equations in the next approximation has a small longitudinal component, and that this apparently small component is sufficient to cancel completely any net energy transfer to a relativistic electron that appears possible if only eq. (1) is used.

Equation (1) describes a continuous-wave laser beam, rather than a pulse.

Equation (1) of [2] is the same as my eq. (1) with the addition of a factor $g(\varphi)$ that describes the laser pulse envelope in time. The new part of what follows is to find a condition on the form of g so that (1) satisfies Maxwell's equations to a good approximation. I will find that the assumption in [2] that $g = \sin^2(\varphi/\varphi_0)$ for $0 < \varphi/\varphi_0 < \pi$ and zero elsewhere is not satisfactory. This oversight is in addition to the erroneous claim in [2] that the longitudinal part of field \mathbf{E} can be neglected.

Before I deal with the issue of acceleration of electrons, I review the derivation of a better approximation to a Gaussian laser pulse, following [3]. The knowledgeable reader might want to skip ahead to eqs. (26) and (27).

¹A small acceleration of electrons “in vacuum” by an intense laser was reported in [2], but this is a subtle effect, not explained in the first approximation presented there.

A key insight of [3] is that the form of eq. (1) can more properly be used for the vector potential \mathbf{A} than for the electric field \mathbf{E} . In general, the divergence of the vector potential need not be zero and there are solutions to Maxwell's equations with nonuniform vector potential that point only along the x -axis.

A second useful insight is that when the wave equation for the vector potential is written in terms of the dimensionless variables

$$\xi = \frac{x}{w_0}, \quad v = \frac{y}{w_0}, \quad \rho^2 = \xi^2 + v^2, \quad \text{and} \quad \varsigma = \frac{z}{z_0}, \quad (4)$$

then a series expansion suggests itself. Namely, the focal region has transverse and longitudinal extent in the ratio

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}. \quad (5)$$

The aspect ratio θ_0 (also the diffraction angle) is typically much less than one, and so can serve as the expansion parameter.

We seek fields that propagate in the $+z$ direction, have limited transverse extent, and for which the vector potential has only an x component. We try

$$\mathbf{A}(\mathbf{r}, t) = \hat{\mathbf{x}}\psi(\mathbf{r})g(\varphi) e^{i\varphi}, \quad (6)$$

where ψ and g vary "slowly." The vector potential must satisfy the free-space wave equation:

$$\nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}. \quad (7)$$

Inserting trial solution (6) into (7) we find that

$$\nabla^2 \psi + 2ik \frac{\partial \psi}{\partial z} \left(1 - \frac{ig'}{g}\right) = 0, \quad (8)$$

where $g' = dg/d\varphi$. Since ψ is a function of \mathbf{r} while g and g' are functions of the phase φ , eq. (8) cannot be satisfied in general. Often the discussion is restricted to the case where $g' = 0$, *i.e.*, to continuous waves. However, we see that to proceed with our description of pulsed beams we must accept the condition that

$$g' \ll g. \quad (9)$$

First, consider the proposal of [2] that $g = \sin^2(\varphi/\varphi_0)$. Then, $g'/g = (2/\varphi_0) \cot(\varphi/\varphi_0)$. Even for the plausible restriction that $\varphi_0 \gg 2$, g'/g blows up at the beginning and end of the pulse (which is defined on the interval $0 \leq \varphi \leq \pi\varphi_0$).

Another popular form for a laser pulse is a Gaussian: $g = \exp[-(\varphi/\varphi_0)^2]$. Then, $g'/g = -2\varphi/\varphi_0^2$ which does not satisfy (9) for $|\varphi| \gtrsim \varphi_0$.

A more appropriate form for a pulsed beam is a hyperbolic secant (as arises in studies of solitons²):

$$g(\varphi) = \operatorname{sech}\left(\frac{\varphi}{\varphi_0}\right). \quad (10)$$

²See, for example, [4].

Then, $g'/g = -(1/\varphi_0) \tanh(\varphi/\varphi_0)$, which is much less than one everywhere for $\varphi_0 \gg 1$.

One might hope that a poor approximation would suffice in regions where the field is weak. But, when the field is probed by a moving electron, the latter spends a long time in the weak-field region, so time integrals such as energy transfer have significant contributions from that region.

Hence, I strongly recommend that future numerical (and analytic) calculations involving laser pulses use form (10).

We now suppose that condition (9) is satisfied, so that (8) can be approximated as

$$\nabla_{\perp}^2 \psi + 4i \frac{\partial \psi}{\partial \varsigma} + \theta_0^2 \frac{\partial^2 \psi}{\partial \varsigma^2} = 0, \quad (11)$$

where

$$\nabla_{\perp}^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial v^2}, \quad (12)$$

in terms of the dimensionless variables introduced in eqs. (4-5). This form suggests the series expansion

$$\psi = \psi_0 + \theta_0^2 \psi_2 + \theta_0^4 \psi_4 + \dots \quad (13)$$

Inserting this into eq. (11) and collecting terms of order θ_0^0 and θ_0^2 , we find

$$\nabla_{\perp}^2 \psi_0 + 4i \frac{\partial \psi_0}{\partial \varsigma} = 0, \quad (14)$$

and

$$\nabla_{\perp}^2 \psi_2 + 4i \frac{\partial \psi_2}{\partial \varsigma} = -\frac{\partial^2 \psi_0}{\partial \varsigma^2}, \quad (15)$$

respectively. Equation (14) can be recognized as the paraxial wave equation whose Gaussian solution was given in eq. (1). That is,

$$\psi_0 = f e^{-f\rho^2}, \quad (16)$$

where

$$f = \frac{-i}{\varsigma - i} = \frac{1 - i\varsigma}{1 + \varsigma^2} = \frac{e^{-i \tan^{-1} \varsigma}}{\sqrt{1 + \varsigma^2}}. \quad (17)$$

In [3], the solution to eq. (15) was cleverly guessed:

$$\psi_2 = \left(\frac{f}{2} - \frac{f^3 \rho^4}{4} \right) \psi_0, \quad (18)$$

although we will not need this here.

We work in the Lorentz gauge (and Gaussian units), so the scalar potential ϕ obeys

$$\frac{\partial \phi}{\partial t} = -c \nabla \cdot \mathbf{A}. \quad (19)$$

Similarly to eq. (6), we suppose that ϕ can be written

$$\phi(\mathbf{r}, t) = \Phi(\mathbf{r}) g(\varphi) e^{i\varphi}. \quad (20)$$

Then,

$$\frac{\partial \phi}{\partial t} = -\omega \phi \left(1 - \frac{ig'}{g}\right) \approx -\omega \phi, \quad (21)$$

when condition (9) is satisfied. In this case,

$$\phi = -\frac{i}{k} \nabla \cdot \mathbf{A}. \quad (22)$$

The electric and magnetic fields can now be deduced from the approximate vector potential

$$\mathbf{A} = \psi_0 g(\varphi) e^{i\varphi} \hat{\mathbf{x}}, \quad (23)$$

via

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{i}{k} \nabla (\nabla \cdot \mathbf{A}) + ik \mathbf{A}, \quad (24)$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (25)$$

again approximating as one the factors $1 - ig'/g$ that arise. The results accurate to order θ_0 are, after dividing out a factor of ik :

$$\begin{aligned} E_x &= \psi_0 g e^{i\varphi}, \\ E_y &= 0, \\ E_z &= \frac{i\theta_0}{2} \frac{\partial \psi_0}{\partial \xi} g e^{i\varphi} = -i\theta_0 f \xi E_x, \end{aligned} \quad (26)$$

$$\begin{aligned} B_x &= 0, \\ B_y &= E_x, \\ B_z &= \frac{i\theta_0}{2} \frac{\partial \psi_0}{\partial v} g e^{i\varphi} = -i\theta_0 f v E_x. \end{aligned} \quad (27)$$

These expression satisfy $\nabla \cdot \mathbf{E} = 0 = \nabla \cdot \mathbf{B}$ plus terms of order θ_0^2 .

After these lengthy preliminaries we are ready to consider vacuum laser acceleration of electrons.

We consider an electron moving at velocity c along the line $x = z\theta$, where θ is a small angle. The electron passes through a laser field given by eqs. (26-27), with $y = 0$ always, and we suppose that it passes the origin at $t = 0$. In terms of the dimensionless variables ξ , ς and ρ , the trajectory is

$$\rho = \xi = \frac{\theta \varsigma}{\theta_0}. \quad (28)$$

The electric field component along the electron's trajectory is

$$E_{\parallel} = E_x \sin \theta + E_z \cos \theta \approx E_x \theta (1 - if\varsigma) = E_x \theta f, \quad (29)$$

for small θ , noting that eqs. (26) and (28) lead to

$$E_z = -i\theta_0 f \xi E_x = -i\theta f \varsigma E_x, \quad (30)$$

and that eq. (17) leads to the identity

$$1 - if\zeta = f. \quad (31)$$

The electron has coordinate z at time $t = z/(c \cos \theta)$ so

$$\varphi = kz - \omega t \approx -\frac{kz\theta^2}{2} = -\frac{\theta^2}{\theta_0^2}\zeta. \quad (32)$$

Then,

$$\begin{aligned} E_x &= E_0 f g e^{-f\rho^2} e^{i\varphi} \\ &\approx E_0 f g e^{-f\zeta^2\theta^2/\theta_0^2} e^{-i\zeta\theta^2/\theta_0^2} = E_0 f g e^{-if\zeta\theta^2/\theta_0^2}, \end{aligned} \quad (33)$$

using eqs. (26), (31) and (32). Inserting this into eq. (29) we have

$$\begin{aligned} E_{\parallel} &= \theta E_0 f^2 g e^{-if\zeta\theta^2/\theta_0^2} \\ &= \frac{d}{d\zeta} \left(\frac{i\theta_0^2}{\theta} E_0 g e^{-if\zeta\theta^2/\theta_0^2} \right), \end{aligned} \quad (34)$$

again neglecting a term in g'/g . That is, the force on the electron along its trajectory can be derived from a potential. The change in energy along the trajectory is then just the change in the potential. However, the potential

$$U = -\frac{i\theta_0^2}{\theta} E_0 g e^{-if\zeta\theta^2/\theta_0^2} \quad (35)$$

has the same value at $\zeta = -\infty$ and $+\infty$, so the (relativistic) electron gains no net energy as it crosses the laser beam.

This is the argument of [1], with the addition that it holds for laser pulses as well as for continuous waves, so long as condition (9) is satisfied – which condition is required if (26) and (27) are to be (approximate) solutions to Maxwell's equations.

Everyone agrees that if one uses E_x from (26) but ignores E_z , a net energy gain will be calculated for a free electron crossing a laser beam. See also [5].

The interesting case of extremely short pulses ($g'/g \approx 1$) has been considered in [6]. The above analysis does not hold in this limit, and significant energy can in principle be transferred from a single-cycle pulse to an electron. However, for a pulse of even a few cycles, the energy transfer is greatly suppressed.

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