“Hidden” Momentum in a Sound Wave?

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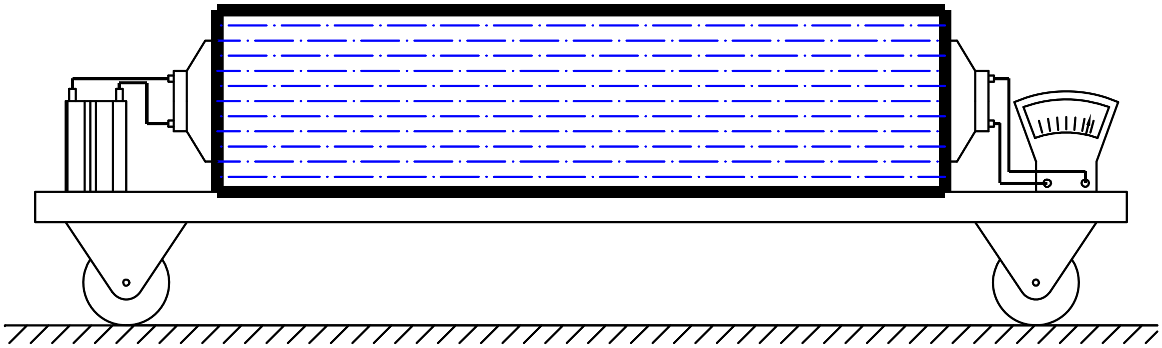
1 Problem

Discuss the momentum of a sound wave. In particular, consider the example sketched below in which a sound wave propagates through a volume of gas at pressure P. The energy of the sound wave is provided by a battery. The entire apparatus is mounted on a horizontal platform that can roll without friction in the direction of the sound wave.



Does this system contain hidden momentum, $\mathbf{P}_{\rm hidden}$, defined (when all velocities are small compared to the speed of light c) by

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}, \tag{1}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total “mass”, U is its total energy, $\mathbf{x}$ is its center of mass/energy, $\mathbf{v} = d\mathbf{x}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, and $\mathbf{v}_b$ is the velocity (field) of its boundary?[1]

2 Solution

The gas molecules appear to have zero average velocity in the rest frame of the pressure vessel, so it appears that they have zero average momentum in this frame.

Of course, the gas molecules exert a force $F = AP$ on a portion of area A of the wall of the pressure vessel, and since force equals the time rate of change of momentum, we can say that the gas can transmit momentum without possessing it [3].

When a sound wave is present, energy is transferred even when there is no net transfer of particles, as in the rest frame of the pressure vessel in the present example. If the rate of energy flow is dE/dt, then the (vector) energy flux is

$$\mathbf{S}_{\rm sound} = \frac{1}{A}\frac{dE}{dt}\,\hat{\mathbf{v}}_{\rm sound} = u_{\rm sound}\,\mathbf{v}_{\rm sound} \tag{2}$$

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[1] The definition (1) was suggested by Daniel Vanzella [1]. See also [2].

where A is the cross sectional area of the pressure vessel, $u_{\rm sound}$ is the density of energy in the sound wave, and $\mathbf{v}_{\rm sound}$ is the velocity of sound with respect to the pressure vessel.

Following the insight of Einstein [4] that energy and mass are related by $E = mc^2$, the energy density $u_{\rm sound}$ is associated with a mass density $u_{\rm sound}/c^2$. Hence, there is a momentum density in the sound wave given by[2]

$$\mathbf{p}_{\rm sound} = \frac{u_{\rm sound}}{c^2}\mathbf{v}_{\rm sound} = \frac{\mathbf{S}_{\rm sound}}{c^2}\,. \tag{3}$$

This momentum density is very small, due to the factor c^2 in the denominator of eq. (3); it is a relativistic effect.

Energy/mass is transferred from the battery to the absorber/meter at the right end of the apparatus, such that the center of mass of the system moves to the right with respect to the pressure vessel. The rate of increase of the mass at the right end is

$$\frac{dM}{dt} = S_{\rm sound}A/c^2\,. \tag{4}$$

This mass appears at distance L to the right of the energy source (battery), where L is the length of the pressure vessel, so the velocity of the center of mass of the system (in the frame of the pressure vessel) is

$$\mathbf{v}_{\rm cm} = \frac{L}{M}\frac{dM}{dt}\hat{\mathbf{S}} = \frac{\mathbf{S}_{\rm sound}AL}{c^2} = \frac{\mathbf{p}_{\rm sound}V}{M} = \frac{\mathbf{P}_{\rm sound}}{M} = \frac{\mathbf{P}}{M}\,, \tag{5}$$

where $V = AL$ is the volume of the pressure vessel, M is the total mass of the system, and

$$\mathbf{P} = \mathbf{P}_{\rm sound} = \mathbf{p}_{\rm sound}V = \frac{\mathbf{S}_{\rm sound}V}{mc^2} = M\mathbf{v}_{\rm cm} \tag{6}$$

is the total momentum associated with the sound wave, and of the entire system.[3,4]

In certain electromechanical systems where mechanical momentum of order $1/c^2$ is present in association with electromagnetic momentum, the term “hidden” momentum has come into use [2, 6, 7, 8, 9, 11, 10, 12, 13, 14, 15, 16]. According to the general definition (1), the present system as a whole does not contain “hidden” momentum in the frame of the pressure vessel, in that the total momentum in the frame of the pressure vessel is $\mathbf{P} = M\mathbf{v}_{\rm cm}$, and the whole system has no boundary.

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[2] The form (3) was anticipated in 1900 by Poincaré [5].

[3] If the system is mounted on frictionless rollers, the system must roll to the left with speed $v_{\rm cm}$ of eq. (5) to leave the position of the center of mass unchanged in the lab frame. As the gas molecules move to the right during the wave motion, they carry greater kinetic energy on average than when they move to the left, in the frame of the pressure vessel. That is, the average velocity of right-moving gas molecules is greater than that of left-moving molecules in this frame. So, in a microscopic view we can say that the gas possesses net Newtonian momentum with respect to the pressure vessel, which momentum points to the right.

[4] Linear longitudinal waves imply zero average velocity for the molecules, so we infer that nonlinear effects must be considered to obtain a full understanding of the momentum density of a sound wave. See, for example, sec. IV of [20]. For sound waves not confined between walls, there can also be a small bulk motion of the medium due to the wave, known as the Stokes drift [17].

Consider now the subsystem consisting only of the gas inside the pressure vessel. The gas is at rest, on average, so $\mathbf{v}_{\rm gas,cm} = 0$. The "hidden" momentum of this subsystem is

$$\begin{aligned}\mathbf{P}_{\rm gas,hidden} &= \mathbf{P}_{\rm gas} - M_{\rm gas}\mathbf{v}_{\rm gas,cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm gas,cm})\,(\mathbf{p}_{\rm gas} - \rho_{\rm gas}\mathbf{v}_b) \cdot d\mathbf{Area} \\ &= \mathbf{P}_{\rm sound} - 0 - \mathbf{p}_{\rm sound}V = 0, \qquad (7)\end{aligned}$$

noting that the velocity of the boundary, $\mathbf{v}_b$, is zero.[5]

3 Comments

It appears that the fluid-dynamics community generally considers that the laws of relativity are not relevant to this branch of physics, and that the momentum density (3) can and should be ignored.[6] For example, a delightful article by McIntyre implies that it is a myth that sound waves possess momentum, even though he acknowledges the content of eqs. (2)-(3) in a footnote [19]. An interesting article by Stone [20] expands on the theme of McIntyre that translational invariance of of a wave with respect to its supporting medium leads to conservation of a quantity that is better called pseudomomentum than Newtonian ($m\mathbf{v}$) momentum. But, Stone argues (in sec. IV of [20]) that there is neither momentum nor pseudomomentum in the present example.[7]

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[5] For an electromechanical example in which electromagnetic energy flows from a battery to a resistor, and the system does contain "hidden" momentum according to definition (1), see [18]. The difference between that example and the present one is that an electromagnetic wave is not directly associated with mechanical momentum, whereas a sound wave is. Both momenta are of order $1/c^2$, and hence are "relativistic" effects.

[6] It is underappreciated that ordinary kinetic energy is a tiny relativistic correction to the total energy of a moving mass: $E_{\rm total} = mc^2/\sqrt{1 - v^2/c^2} \approx mc^2(1 + v^2/2c^2) = mc^2 + mv^2/2 = E_{\rm rest\ mass} + E_{\rm kinetic}$. Thus, all discussions of kinetic energy in classical mechanics concern tiny relativistic corrections of order $1/c^2$, in which, however, the factor $1/c^2$ is not explicitly displayed.

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