A Simplified View of the Higgs/Yukawa Mechanism

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1 Problem

The recent apparent observation [1, 2] of the Higgs boson [3, 4, 5, 6] has led to much interest in the popular claim that the Higgs boson (or better, the Higgs field) "gives mass" to those elementary particles that have it. Discuss how the de Broglie relations [7] for "matter waves" lead to a wave equation with a term that depends on mass, and that interpretation of this term as due to the interaction of the particle/wave with short-range "background" field "explains" the origin of mass. Compare with (quasi)electrons in an electromagnetic wave.

2 Solution

2.1 A Wave Equation for Matter Waves

In Maxwell's theory [8] a scalar component ψ of the electromagnetic field obeys the wave equation (in vacuum),

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \tag{1}$$

where c is the speed of light in vacuum. Then, if the field component is a plane wave of the form $\psi(\mathbf{x},t) = \psi_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, the wave vector $\mathbf{k}$ and the angular frequency ω are related by the dispersion relation, obtained by inserting the plane-wave form into the wave equation,

$$k^2 = \frac{\omega^2}{c^2}\,. \tag{2}$$

In 1905, Einstein postulated that electromagnetic waves were quantized [9], and that the quanta (photons) have energy E given by

$$E = \hbar\omega, \tag{3}$$

where $\hbar = h/2\pi$ is Planck's constant [10]. In 1909, Einstein noted [11] that photons also carry momentum P but did not explicitly state that

$$P = \frac{\hbar\omega}{c} = \hbar k, \tag{4}$$

such that the photon's energy and momentum are related by[1]

$$E = cP. \tag{5}$$

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[1] The relation (5) is implicit in sec. 792 of Maxwell's *Treatise* [8], which noted that the radiation pressure p of an electromagnetic wave equals its energy density E/Vol: $p = F/A = dP/A\,dt = cP/\text{Vol} = E/\text{Vol}$.

In 1923, de Broglie postulated [12] that massive particles could be represented by waves in the quantum theory, where the energy of a particle/wave is given by eq. (3); in his Ph.D. thesis [7] he argued that the momentum is given by eq. (4), while also being related by

$$E^2 = c^2 P^2 + m^2 c^4, \tag{6}$$

$$\omega^2 = k^2 c^2 + \frac{m^2 c^4}{\hbar^2} = k^2 c^2 + \frac{c^2}{\lambda\!\!\!^{-2}}, \tag{7}$$

where m is the rest mass of the particle,[2] and

$$\lambda\!\!\!^{-} = \frac{\hbar}{mc} \tag{8}$$

is the so-called reduced Compton wavelength of the particle/wave.

In 1925, De Broglie noted [15] that plane waves of light in a medium of refractive index $n = \sqrt{1 + \Delta^2}$ obey the wave equation

$$\nabla^2 \psi = \frac{n^2}{c^2} \frac{\partial^2 \psi}{\partial t^2}, \tag{9}$$

for which the dispersion relation is

$$k^2 c^2 = \omega^2 + (\omega \Delta)^2. \tag{10}$$

Here, the (mass?) term $(\omega\Delta)^2$ has the wrong sign for photons inside a refractive medium to behave as expected for matter waves with real, positive mass.[3,4,5]

A suitable wavefunction for scalar matter waves was given in 1926 by Klein, Fock and Gordon [18, 19, 20],[6]

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \psi = 0, \tag{11}$$

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[2] Although the relations $E = \gamma mc^2$ and $\mathbf{P} = \gamma m \mathbf{v}$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$, are implicit in eq. (11) of Einstein's review of 1908 [13], and lead to eq. (6), this important relation seems not to have been explicitly stated until 1914 in [14].

[3] Photons inside a refractive medium are now considered to be virtual particles with negative mass squared, which notion became acceptable only much later than 1925.

[4] The index of refraction of a medium can be thought of as arising from the scattering of the incident electromagnetic wave (with velocity c in vacuum) off atoms in the medium. See, for example, chap. 31 of [16]. This indicates that the popular model of the Higgs mechanism as due to elastic scattering of nominally massless particles off "ping-pong balls" leads to negative, rather than positive, mass squared.

[5] A classical electromagnetic example whose virtual photons have positive mass is propagtion in a rectangular waveguide (well known only after 1936 [17]). For example, the TE_{10} mode in a guide of cross section $a \times b$ has fields that can be written (in Gaussian units) as $E_x = 0$, $E_y = E_0 \sin \pi x/a$, $E_z = 0$, $H_x = -(ck_g/\omega)E_y$, $H_y = 0$, $H_z = -i(\pi c/a\omega)E_0 \cos \pi x/a$, with the dispersion relation $\omega^2 = k_g^2 c^2 + (\pi c/a)^2$ corresponding to mass $m = \hbar\pi/ac$. The group velocity is $v_g = d\omega/dk_g = c^2 k_g/\omega$ $(= c^2/v_{\mathrm{phase}}) < c$. The average energy density in the wave is $\langle u \rangle = (|E_y|^2 + |H_x|^2 + |H_z|^2)/32\pi = E_0^2/16\pi$, and the average momentum density (along the guide axis) is $\langle p_z \rangle = |E_y| \, |H_x| / 16\pi c = E_0^2 k_g / 16\pi\omega = \langle u \rangle \, v_g/c^2$, as expected for a particle with (relativistic) velocity v_g. Here, one might say that the guide walls slow down the wave and give it mass, which is not a good analog of the Higgs mechanism.

[6] Apparently, Schrödinger considered eq. (11) in 1925 but did not publish it, choosing instead to develop a nonrelativistic version.

for which the dispersion relation is given by eq. (7). Formally, simply adding the third term in eq. (11) "gives mass" to the wavefunctions ψ that are described by this wave equation. Initially, people did not seek an explanation of this "mass" term as arising from some other physical process.

2.2 Yukawa and Meson Theory

An important insight drawn from the wave equation (11) was by Yukawa in 1935 [21], that if one sought to explain a short-range interaction (such as the nuclear force) with characteristic length scale $\lambda\!\!\!^{-}$, then this interaction may be due to the exchange of a particle/wave of mass $m = \hbar/c\lambda\!\!\!^{-}$. This led to the prediction of a "meson" with mass about 200 times that of the electron, which should exist inside nuclei, and should be able to go free as a result of sufficiently energetic nuclear collisions.

Yukawa's meson was not discovered until 1947 [22, 23, 24], although the physics community was "distracted" in the late 1930's by the discovery of a new particle [25, 26], now called the muon, with mass almost exactly as Yukawa predicted, but which does not participate in nuclear interactions. In this context, attention centered for many years on understanding elementary-particle interactions, and on the discovery of new particles, rather than on explanation of where mass "comes from". However, in 1950 Yukawa [27] made a "flip" of his theory, whereby he conjectured that the mass of one type of particle/wave might be related to its short-range interaction with some other field.[7]

2.3 A Mechanical Analogy

In a mechanical analogy, consider string of mass density ρ per unit length stretched under tension T along the x-axis. Then the equation of motion for small displacements in the y-direction is

$$\rho\ddot{y} = Ty'', \tag{12}$$

and the wave velocity is

$$v = \sqrt{\frac{T}{\rho}}\,. \tag{13}$$

If the string were attached to the x-axis everywhere along its length by a set of springs, resulting in spring constant K per unit length for displacements in the y-direction, the wave equation would become

$$\rho\ddot{y} = Ty'' - Ky, \tag{14}$$

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[7] Yukawa's vision, later incorporated in the "Higgs mechanism", is that "mass" is a result of an interaction described by a relativistic wave equation. Nonetheless, there exists a debate on whether or not the term "relativistic mass" should be permitted in physics discussions. Some of the history of this (pointless and therefore endless) debate is given in chap. 2 of [28].

for which the dispersion relation is

$$\omega^2 = k^2 v^2 + \frac{K}{\rho}\,, \tag{15}$$

which has the same form as eq. (7) for matter waves. That is, quanta of the waves on the string have mass proportional to $\sqrt{K/\rho}$, which quanta are massless without the set of springs, while nonzero when they are present and interact with the string.

Hence, we arrive at a vision that mass can be "given" to massless wave/particles if the amplitude of these waves is constrained to stay near zero at any point by some kind of spring-like interaction with the "vacuum".

In a wave/field view, this interaction should be associated with some field with short-range interaction that exists everywhere in space and time, and whose primarily manifestation is to "give mass". In the quantum view (following the spirit of Yukawa's original suggestion [21]), that short-range field is associated with massive quanta, now called Higgs bosons.

This simplified version of the "Higgs mechanism" leaves open many questions, such as the nature of the mass-giving field, and why different particle/waves have different masses.

2.3.1 Related Example: Car on a "Washboard" Road

As a mechanical analogy of a quasiparticle that "gets mass" from an interaction with a "background field", consider a car of nominal mass m and initial velocity v_0 that coasts onto a "washboard" road with vertical undulations of amplitude A and period λ along the road (see [29], particularly the Appendix). If the wheels of the car are connected to the body via shock absorbers of total spring constant k, and we ignore the damping present in real shock absorbers, then the car oscillates vertically according to

$$y(t) = A\frac{\omega_0^2 \sin\omega t}{\omega_0^2 - \omega^2}\,, \tag{16}$$

where $\omega_0 = \sqrt{k/m}$, $\omega = 2\pi v/\lambda$, and v is the horizontal speed of the car when on the "washboard" road, assuming that a steady state could be somehow achieved without damping. Nonzero energy,

$$U_{\rm osc} = k\,\langle y(t)\rangle^2 = \frac{kA^2}{2}\frac{\omega_0^4}{(\omega_0^2 - \omega^2)^2} < \frac{mv_0^2}{2}\,, \tag{17}$$

is associated with these oscillations, so conservation of energy implies that the car slows down on the washboard road to velocity v given by

$$\frac{mv_0^2}{2} = \frac{mv^2}{2} + U_{\rm osc}. \tag{18}$$

If we take a view that ignores the tranverse oscillations and emphasizes only the longitudinal velocity of the car, we might say that when on the washboard road it has effective mass $\overline{m}$ related by

$$\frac{mv_0^2}{2} = \frac{\overline{m}v^2}{2}\,, \tag{19}$$

and hence,

$$\overline{m} = m + \frac{2U_{\rm osc}}{v^2} = m + \frac{2U_{\rm osc}}{mv_0^2 - 2U_{\rm osc}}\,. \tag{20}$$

We might now say that the car has become a "quasicar" with effective mass $\overline{m}$. This "quasicar" has been "given" (additional) mass by its interaction with the "washboard" road, which is a kind of a "background field".

In the quantum realm, we say that the Higgs background field "gives mass" to otherwise massless elementary (quasi)particles, by a mechanism somewhat analogous to the case of a car on a "washboard" road.

2.4 Digression: Electromagnetic Self-Energy/Mass

The earliest suggestion of a field-theory origin of mass may be the vision of J.J. Thomson in 1881 [30] that the mass of an electrically charged particle is due in part (or all) to effects of its own electromagnetic field (anticipating aspects of Einstein's relation $E = mc^2$ [31] by many years). The famous issue with this vision is that if the particle is small enough, the electromagnetic self-mass becomes arbitrarily large, such that this view works too well in explaining the origin of mass.

The issue of apparently infinite self-masses associated with "pointlike" charged particles persists in quantum theory, where it was "swept under the rug" by the (highly successful) mathematical technique of renormalization.[8] A price of this success is that one is left with no quantitative understanding of the contribution to the mass of an electrically charged particle/wave due to its own electric field, which leaves room for other explanations of mass.

The Higgs background field contributes to the self-energy/mass of its quanta, the Higgs bosons. We cannot compute the mass of the Higgs bosons, but we can conclude that it must be nonzero (and that the Higgs mechanism is renormalizable), and hence the Higgs field is short range, as stated above without justification.

2.5 Mass Shift of a Charged Particle in a Plane Electromagnetic Wave

Another vision of an origin of mass from the year 1935 is due to Volkov [34, 35], who noted that a Dirac electron which propagates inside a strong (classical) electromagnetic plane wave has an effective mass $\overline{m}$ given by

$$\overline{m} = m\sqrt{1 + \eta^2}, \tag{21}$$

where m is the mass of the electron in zero field, and the dimensionless, relativistic invariant[9,10] η is given, for a circularly polarized electromagnetic plane wave of electric field

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[8] The first work on renormalization in quantum theory is apparently [32]. For commentary on this pioneering work, see [33].

[9] We use a metric such that $A_\mu A^\mu = A_0^2 - \mathbf{A}^2$, and consider the vector potential A_μ in the Lorenz gauge.

[10] Many texts ignore the existence of this invariant of the electromagnetic field, emphasizing only that $E^2 - B^2$ and $\mathbf{E} \cdot \mathbf{B}$ are invariants.

amplitude E_0, by

$$\eta = \frac{e}{mc^2}\sqrt{-A_\mu A^\mu} = \frac{eE_0}{m\omega c} \tag{22}$$

where $-e$ is the charge of an electron. Volkov's discussion of an electron "dressed" by a background electromagnetic wave was the first description of what is now called a quasiparticle, which concept came to be developed much more in condensed-matter physics than in elementary-particle physics prior to 1960.

When the electromagnetic wave is circularly polarized, and the average motion of the electron is along the direction of the wave, the electron's motion is a helix of radius r.[11] We can introduce a transverse velocity $\beta_\perp$ and a transverse relativistic factor $\gamma_\perp$ through an analysis in the average rest frame of the electron (the * frame, with $r^\star = r$, $\beta_\perp^\star = \beta_\perp$, $\gamma_\perp^\star = \gamma_\perp$):

$$eE_0^\star = \gamma_\perp^\star m\omega^{\star 2} r^\star = \gamma_\perp^\star \beta_\perp^\star m\omega^\star c, \tag{23}$$

where

$$\beta_\perp = \beta_\perp^\star = \frac{v_\perp^\star}{c} = \frac{\omega^\star r^\star}{c}, \qquad \gamma_\perp^\star = \left(1 - \beta_\perp^{star2}\right)^{-1/2} = \gamma_\perp = \left(1 - \beta_\perp^2\right)^{-1/2}. \tag{24}$$

Therefore, the invariant parameter η of eq. (22) can be written as

$$\eta = \frac{eE_0}{m\omega c} = \frac{eE_o^\star}{m\omega^\star c} = \gamma_\perp^\star \beta_\perp^\star = \gamma_\perp \beta_\perp \qquad \text{and} \qquad \gamma_\perp = \left(1 + \eta^2\right)^{1/2}. \tag{25}$$

In the * frame (in which the electron is at rest on average, while moving in a small circle with velocity $v_\perp = \beta_\perp c$), it has energy $\gamma_\perp mc^2$, and we say that it has acquired an "effective mass" related by

$$\overline{m} = m\gamma_\perp = m\sqrt{1+\eta^2}, \tag{26}$$

in a description where the transverse motion is averaged over.

In a quantum description one does not refer to the classical path of the transverse motion of the electron in the wave field, but rather only the longitudinal motion is described (by the Volkov solutions to the Dirac equation [35]). The kinematic character of the quantum quasiparticle can be summarized by writing the four-vector of the (quasi)electron as

$$\overline{p}_\mu = p_\mu + \kappa\omega_\mu, \tag{27}$$

where p_μ is the four-vector of the electron in the absence of the wave, ω_μ is the four-vector of a wave photon, and $\kappa = \eta^2 m^2 c^4 / 2(p \cdot \omega)$. The latter relation follows from setting $\overline{p}^2 = \overline{m}^2 c^4$. Loosely speaking, an electron in a wave field has absorbed (been "dressed" by) κ wave photons, which shifts upwards the effective mass of the (quasi)electron.

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[11] Motion of a charged particle in a linearly polarized electromagnetic wave is technically more complicated than that in a circularly polarized wave. See, for example, sec. 48 of [36], [37], and [38] for the simpler case of $\eta \ll 1$.

The mass-shift effect described here (for electrons in a wave field) has never been detected directly in an experiment (while quasielectrons in solids are well known as plasmons, polaritons, etc.), although its indirect effect was seen in an experiment in which electron-positron pairs were created in an intense laser beam probed by high-energy electrons [39]. In particular, if an electron or positron is created in a strong wave field, its invariant mass is immediately $\overline{m}$, which relaxes to m only when the particle leaves the wave field. That "heavy" electrons and positrons must be created in a strong field affects details of the production mechanism.[12]

The close conceptual relation between the effective mass of electromagnetic quasiparticles and the Higgs (quasiparticle) mechanism led one of its originators, Kibble, to write on this theme in 1966 [43].

2.6 Higgs *et al.*

A decade after Yukawa's suggestion that some kind of short-range "background" field might be responsible for all mass of particle/waves (which are to be considered as quasiparticles in the background field), this theme was revived by Nambu [44], who noted that the dispersion relation (11) for massive particle/waves is formally equivalent to a description of the band gap of a semiconductor in **k**-space. Nambu's further remarked on the possible relation of quasiparticles to the gauge invariance of the theory, which led to rapid developments of this theme in elementary-particle theory [45] and in condensed matter theory [46], followed in 1964 by the works of Higgs [3, 4], of Englert and Brout [5], and of Guralnik, Hagen and Kibble [6].

These abstract field-theoretic developments were applied in 1967 by Weinberg [47] and by Salam [48] to an (electroweak gauge) theory that unified the electromagnetic and so-called weak interactions as being due to four related fields with electrically neutral quanta called photons and Z^0 bosons, and electrically charged quanta called $W^\pm$ bosons. In principle, all four of these fields would imply long-range interactions via massless quanta, but the interactions of these fields with a background Higgs field led to a prediction that only the photon remains massless,[13] while the masses of the Z^0 and the $W^\pm$ bosons are in the ratio of a certain function of weak-interaction coupling constants.

This is the most detailed prediction to emerge from consideration of the "Higgs mechanism", and it has been well verified experimentally. This success lends credence to the claim that the interaction of the background Higgs field with elementary particles "gives" (most of) them mass, although there is no prediction as to the amount of mass so given (except for the special case of the four electroweak gauge bosons).

Thus, the story of the origin of mass is far from complete, but the insights of the

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[12] Other comments by the author on classical effects of charged particles in intense electromagnetic fields include [40, 41]. For example, if an electron initially at rest, $p_\mu = (mc^2, 0, 0, 0)$, is overtaken by a plane electromagnetic wave, $\omega_\mu = \hbar\omega(1, 0, 0, 1)$, then $\kappa = \eta^2 mc^2/2\hbar\omega$, $\overline{p}_\mu = (mc^2(1 + \eta^2/2), 0, 0, \eta^2 mc/2)$, so the electron inside the plane wave has longitudinal velocity $v_\parallel = \eta^2 c/(2 + \eta^2)$, although it loses this velocity if/when it exits the plane wave. Non-plane electromagnetic waves in vacuum can impart net energy (accelerate) electrons, although it is difficult to obtain significant net energy transfer by this process [42].

[13] The other "known" fundamental bosons, the gluons that mediate the strong/nuclear interaction, and the graviton, also not given mass by the Higgs mechanism.

“Higgs/Yukawa mechanism” are a step towards greater understanding.

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