# A Josephson Junction

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## 1 Problem

A Josephson junction is formed when two superconducting wires are separated by an insulating gap of capacitance $C$. The quantum states $\psi_i$, $i = 1, 2$ of the two wires can be characterized by the numbers $n_i$ of Cooper pairs (charge $2e$) and the phases $\theta_i$, such that $\psi_i = \sqrt{n_i} e^{i\theta_i}$ (Ginzburg-Landau approximation). The (small) amplitude that a pair tunnel across a narrow insulating barrier is $-E_J/n_0$, where $n_0 = n_1 + n_2$ and $E_J$ is the so-called Josephson energy.

The interesting physics is expressed in terms of the differences $n \equiv n_2 - n_1$ and $\phi \equiv \theta_2 - \theta_1$. Consider a junction where $n_1 \approx n_2 \approx n_0/2$.

1. Deduce equations of motion for $n$ and $\phi$.
2. Show that when the relative phase $\phi$ is nonzero a dc current $J_s$ flows across the junction. What is the maximum possible value of $J_s$?
3. Deduce the natural oscillation frequency $\omega_J$ of the junnction (called the Josephson plasma frequency). What is the equilibrium state about which the system oscillates?
4. Suppose a dc voltage $V$ is applied across the junction by a battery. Show that this leads to an oscillating pair current across the junction. Give an expression for the angular frequency $\omega$ of this oscillation.

## 2 Solution

This problem was suggested by N.P. Ong based on some notes by P.W. Anderson.

*[There appear to be some factor of 2 errors below.]*

1. When there exists a nonzero difference $n$ between the number of pairs of charge $-2e$, where $e > 0$, on the two sides of the junction, there is net charge $-ne$ on side 2 and net charge $ne$ on side 1. Hence, a voltage difference $en/C$ arises, where the voltage on side 1 is higher than that on side 2 if $n = n_2 - n_1 > 0$. Taking the zero of the voltage to be at the center of the junction, the electrostatic energy of a Cooper pair of charge $-2e$ on side 2 is $ne^2/C$, and that of a pair on side 1 is $-ne^2/C$.

   [The total electrostatic energy is $C\Delta V^2/2 = Q^2/2C = (ne)^2/2C$.]

   The equations of motion for a pair in the two-state system $\{1, 2\}$ are

$$i\hbar \frac{d\psi_1}{dt} = U_1\psi_1 - \frac{E_J}{n_0}\psi_2 = -\frac{ne^2}{C}\psi_1 - \frac{E_J}{n_0}\psi_2, \tag{1}$$

$$i\hbar \frac{d\psi_2}{dt} = U_2\psi_2 - \frac{E_J}{n_0}\psi_1 = \frac{ne^2}{C}\psi_2 - \frac{E_J}{n_0}\psi_1. \tag{2}$$

Using $\psi_i = \sqrt{n_i}e^{i\theta_i}$ we find

$$i\hbar\left(\frac{\dot{n}_1}{2\sqrt{n_1}}e^{i\theta_1} + i\dot{\theta}_1\sqrt{n_1}e^{i\theta_1}\right) = -\frac{ne^2}{C}\sqrt{n_1}e^{i\theta_1} - \frac{E_J}{n_0}\sqrt{n_2}e^{i\theta_2}, \tag{3}$$

$$i\hbar\left(\frac{\dot{n}_2}{2\sqrt{n_2}}e^{i\theta_2} + i\dot{\theta}_2\sqrt{n_2}e^{i\theta_2}\right) = \frac{ne^2}{C}\sqrt{n_2}e^{i\theta_2} - \frac{E_J}{n_0}\sqrt{n_1}e^{i\theta_1}, \tag{4}$$

or

$$i\hbar\frac{\dot{n}_1}{2} - \hbar n_1\dot{\theta}_1 = -\frac{ne^2}{C}n_1 - \frac{E_J}{n_0}\sqrt{n_1 n_2}e^{i\phi}, \tag{5}$$

$$i\hbar\frac{\dot{n}_2}{2} - \hbar n_2\dot{\theta}_2 = \frac{ne^2}{C}n_2 - \frac{E_J}{n_0}\sqrt{n_1 n_2}e^{-i\phi}, \tag{6}$$

Taking real and imaginary parts,

$$\dot{\theta}_1 = \frac{ne^2}{\hbar C} + \frac{E_J}{\hbar n_0}\sqrt{\frac{n_2}{n_1}}\cos\phi, \tag{7}$$

$$\dot{n}_1 = -\frac{E_J}{\hbar n_0}\sqrt{n_1 n_2}\sin\phi, \tag{8}$$

$$\dot{\theta}_2 = -\frac{ne^2}{\hbar C} + \frac{E_J}{\hbar n_0}\sqrt{\frac{n_1}{n_2}}\cos\phi, \tag{9}$$

$$\dot{n}_2 = \frac{E_J}{\hbar n_0}\sqrt{n_1 n_2}\sin\phi. \tag{10}$$

Taking differences, we find the equations for $n$ and $\phi$,

$$\dot{\phi} = \dot{\theta}_2 - \dot{\theta}_1 = -\frac{2ne^2}{\hbar C} - \frac{E_J}{\hbar n_0}\left(\sqrt{\frac{n_2}{n_1}} - \sqrt{\frac{n_1}{n_2}}\right)\cos\phi \approx -\frac{2ne^2}{\hbar C}, \tag{11}$$

$$\dot{n} = \dot{n}_2 - \dot{n}_1 = +\frac{2E_J}{\hbar n_0}\sqrt{n_1 n_2}\sin\phi \approx \frac{E_J}{\hbar}\sin\phi, \tag{12}$$

noting that $n_1 \approx n_2 \approx n_0/2$. Taking the sums, we find that $n_0 =$ constant, and an amusing but not very relevant equation for the time dependence of $\theta_1 + \theta_2$.

2. We identify a pair (electrical) current from side 1 to side 2 as

$$J_s = (-2e)\frac{\dot{n}}{2} = -\frac{eE_J}{\hbar}\sin\phi \equiv J_0\sin\phi, \tag{13}$$

using eq. (12), where the maximum current is

$$J_0 = \frac{eE_J}{\hbar} = \frac{2\pi eE_J}{h} = \frac{\pi E_J}{\varphi_0}, \tag{14}$$

where $\varphi_0 = h/2e$ is the flux quantum.

3. To exhibit oscillatory behavior we use eq. (12) in the derivative of eq. (11) to find

$$\ddot{\phi} \approx -\frac{2e^2 E_J}{\hbar^2 C} \sin\phi, \tag{15}$$

If $E_J$ is positive, then there are oscillations about $\phi = 0$ whose angular frequency is given by

$$\omega_J = \sqrt{\frac{2e^2 E_J}{\hbar^2 C}} \tag{16}$$

for small amplitudes.

If $E_J$ is negative, then there are oscillations about $\phi = \pi$, since $\sin(\pi - \phi) = \sin\phi$ while $d^2(\pi - \phi)/dt^2 = -\ddot{\phi}$. The frequency of oscillation is again given by eq. (16), now using $|E_J|$.

The form of eq. (15) suggests that $-E_J \sin\phi$ be considered as a generalized force with respect to coordinate $\phi$. That is

$$F_\phi \propto -E_J \sin\phi = -\frac{\partial U}{\partial \phi}, \tag{17}$$

so that

$$U \propto -E_J \cos\phi + ... \tag{18}$$

This further suggests that we reconsider the system in terms of coordinates $n$ and $\phi$. A Hamiltonian in terms of these coordinates would include the electrostatic energy $(ne)^2/2C$ as well as the tunneling energy $-E_J \cos\phi$. We recall that $E_J/n_0$ was the amplitude for one out of the total of $n_0$ pairs to tunnel across the junction, so $E_J$ is the normalized tunneling energy for the whole system. Then, a suitable Hamiltonian is

$$H = \frac{(ne)^2}{2C} - E_J \cos\phi. \tag{19}$$

We could then deduce the equations of motion for $n$ and $\phi$ from $H$ via

$$\dot{n} = \frac{i}{\hbar}[H, n] = \frac{i}{\hbar}\frac{\partial H}{\partial \phi}[\phi, n], \qquad \dot{\phi} = \frac{i}{\hbar}[H, \phi] = \frac{i}{\hbar}\frac{\partial H}{\partial n}[n, \phi], \tag{20}$$

provided we know the commutation relation $[\phi, n]$. Working backwards (or otherwise?) we find that we need

$$[n, \phi] = i. \tag{21}$$

Thus,

$$\dot{n} = \frac{i}{\hbar}(E_J \sin\phi)(-i) = \frac{E_J \sin\phi}{\hbar}, \tag{22}$$

which agrees with eq. (12), and

$$\dot{\phi} = \frac{i}{\hbar}\frac{ne^2}{C}(i) = -\frac{ne^2}{\hbar C} \tag{23}$$

which more or less agrees with eq. (11).

4. If a voltage $V = V_1 - V_2$ is applied across the junction, we expect charge $Q_1 = VC = (-2e)(-n/2) = en$ to be held on side 1, and the negative of this on side 2. Then, eq. (1) becomes

$$\dot{\phi} \approx -\frac{2eV}{\hbar} \equiv -\omega\,, \tag{24}$$

where

$$\omega = \frac{2eV}{\hbar}\,. \tag{25}$$

Equation (24) integrates to $\phi = -\omega t$.

The battery holds the charge difference across the junction fixed at $en = VC$, but can be a source of sink of charge such that a current can flow in the circuit. The claim is that in the present case, the current is given by eq. (13), so

$$J_s = -J_0 \sin \omega t. \tag{26}$$

A possible argument (Feynman Red Book III, p. 21-15) is that as $\phi$ moves off zero, eq. (13) describes the resulting current across the junction in an isolated system. For a system hooked to a battery, this current flows through the entire system, while the numbers of pairs $n_1$ and $n_2$ remain fixed at $\mp n/2 = \mp VC/2e$.

Accepting this argument, the DC voltage of the battery results in an AC current in the circuit of angular frequency (25).