# Intensity, Brightness and Étendue
# of an Aperture Lamp

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## 1  Problem

An aperture lamp is a variant of a fluorescent bulb in which the phosphor[1] is deposited on a reflecting substrate (which acts as a kind of optical insulator), and both of these exist over only a portion of the azimuth of the cylindrical glass housing, as sketched below.
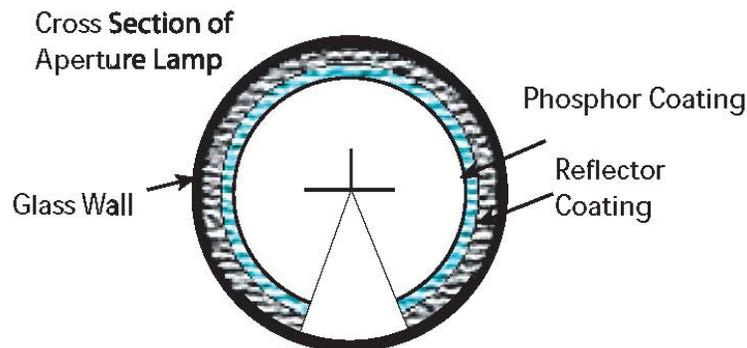


Phosphor molecules are excited by ultraviolet light emitted during the steady electrical discharge of the low-pressure gas inside the lamp, and subsequently de-excite via emission of visible light. Some of this light is absorbed and re-emitted by the phosphor. In this problem, assume that there are no losses in this absorption/re-emission.

Compare the intensity, brightness and étendue of the light from an ordinary fluorescent lamp (with no reflector and phosphor over the full azimuth) with that of an aperture lamp, with small angular aperture $\Delta\phi$, with the same power output in the visible light. You may assume that the phosphor surface emits radiation according to Lambert's cosine law [2].[2]

Note that the principle of this problem applies equally well to a lamp consisting of an array of light-emitting diodes (which also absorb and re-emit light with little loss).[3,4]

Show that the effect of the phosphor/reflector (optical insulator) is to increase the intensity/brightness/temperature of the light inside the lamp (and the light emitted by it) for a

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[1]The phosphors are typically metal oxides with a $PO_4$ radical, often with rare-earth elements. The first variant of an aperture lamp may be from 1936 [1].

[2]See, for example, sec. 4.8 of [3].

[3]The principle also applies to an incandescent lamp in which a reflector (mirrorlike or diffuse) directs some of the light back onto the filament. The small size of tungsten filaments makes this a small effect, which apparently went unnoticed until (2000) [4]. Subsequently it was noted that the principle (there called "light recycling") applies to extended light sources [5], such as LEDs, and that these sources can also serve as diffuse reflectors as in the fluorescent aperture lamp.

[4]A variant of "light recyling" is important in improving the signal of gravitational waves in giant optical interferometers [6, 7, 8].

given input power.[5]

# 2 Solution

In this problem we suppose the lamp is infinitely long and has radius $a$, its wall and coatings are infinitely thin, and that its behavior can be described in only 2 spatial dimensions.

## 2.1 Intensity

If the luminous power output of the lamp is $P_0$, then an "ordinary", azimuthally symmetric lamp has intensity $I$ on a cylindrical screen at radius $r > a$ given by

$$I_{\text{ordinary}}(r > a) = \frac{dP_{\text{received}}}{ds} = \frac{P_0}{2\pi r} , \tag{1}$$

where $s$ measures distance in the azimuthal direction on screen.

For a lossless aperture lamp where the aperture has small azimuthal extent $\Delta\phi$, all of the luminous power exits through the aperture, and the intensity in the aperture is

$$I_{\text{aperture}}(r = a) = \frac{P_0}{a\Delta\phi} = N I_{\text{ordinary}}(r = a), \qquad \text{where} \qquad N = \frac{2\pi}{\Delta\phi} . \tag{2}$$

*On average, a photon makes $N = 2\pi/\Delta\phi$ bounces (i.e., is absorbed and re-emitted $N$ times) before emerging through the aperture, as confirmed on the next page.*

We now calculate the angular dependence $dI_{\text{aperture}}(r = a, \phi)/d\theta$ of the intensity (2) that passes through the aperture, as observed far from the lamp at angle $\theta$ to the normal through the aperture.

All the light that emerges in the interval $d\theta$ about angle $\theta$ was last emitted from arc $ds$ on the glowing surface of the lamp at radius $a$, as shown in the figure below. That arc is at distance $b = 2a\cos\theta$ from the aperture (noting the isosceles triangle with sides $a$, $a$ and $b$ with equal interior angles $\theta$). The aperture has arc length $\Delta s' = a\Delta\phi$, and subtends angle $d\theta$ with respect to the glowing arc $ds$. The projection $ds'' = b\,d\theta = 2a\,d\theta\cos\theta$ of the aperture onto a circle of radius $b$ about the arc $ds$ is also related by $ds'' = \Delta s'\cos\theta = a\Delta\phi\cos\theta$, such that $d\theta = \Delta\phi/2$.[6]



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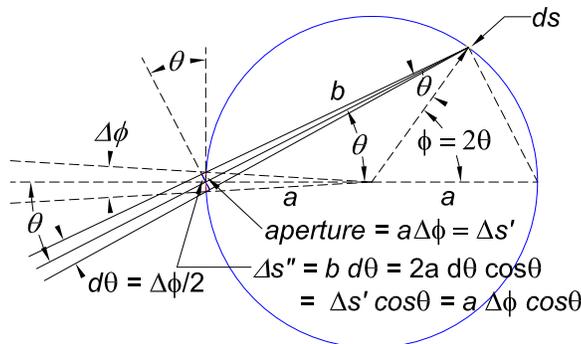
[5]The effect of the "light recycling" by the phosphor/reflector is similar to that of insulation of the walls of a house in permitting a higher internal temperature for a given heat input.

[6]This result is not related to the fact that the arc $ds$ is at angle $\phi = \theta/2$ with respect to the inward normal through the aperture.

Inside an aperture lamp with a small aperture, the power $dP_{\text{phosphor}}/ds$ emitted by any arc segment $ds$ is the same, but it is larger than $P_0\,ds/2\pi a$ because the light bounces around inside the lamp before exiting. We assume here that the angular distribution $d^2 P_{\text{phosphor}}/ds\,d\theta$ of the emitted light is proportional to $\cos\theta$ over the angular range $-\pi/2 < \theta < \pi/2$ with respect to the inward normal to the arc segment $ds$, i.e.,[7]

$$\frac{d^2 P_{\text{phosphor}}}{ds\,d\theta} = \frac{\cos\theta}{2}\frac{dP_{\text{phosphor}}}{ds} \qquad \text{(Lambert's law)}. \tag{3}$$

Hence, the power that passes through the aperture after last being emitted on arc segment $ds$ at angle $\theta$ is

$$dP_{\text{aperture}}(r = a, \theta) = \frac{d^2 P_{\text{phosphor}}}{ds\,d\theta}\,ds\,d\theta = \frac{\cos\theta}{2}\frac{dP_{\text{phosphor}}}{ds}\,ds\,\frac{\Delta\phi}{2}, \tag{4}$$

which also follows Lambert's cosine law. The total power that passes through the aperture is then

$$\begin{aligned}
P_{\text{aperture}} &= \int dP_{\text{aperture}} = \frac{\Delta\phi}{4}\frac{dP_{\text{phosphor}}}{ds}\int \cos\theta\,ds \\
&= \frac{\Delta\phi}{4}\frac{dP_{\text{phosphor}}}{ds}\int_{-\pi}^{\pi}\cos(\theta = \phi/2)\,a\,d\phi = \frac{dP_{\text{phosphor}}}{ds}\,a\Delta\phi = P_0,
\end{aligned} \tag{5}$$

and hence,

$$\frac{dP_{\text{phosphor}}}{ds} = \frac{P_0}{a\Delta\phi}, \qquad \text{and} \qquad P_{\text{phosphor}} = \int \frac{dP_{\text{phosphor}}}{ds}\,ds = \frac{2\pi}{\Delta\phi}P_0 = NP_0. \tag{6}$$

The result (6), that the total power emitted by the phosphor in an aperture lamp is larger by the factor $N = 2\pi/\Delta\phi$ than that in an "ordinary" lamp, is consistent with eq. (2), yet the "wall-plug" power into the two lamps is the same. This does not violate conservation of energy, in that the light which does not immediately pass through the aperture is stored within the lamp via successive absorption and re-emission until it passes through later. This "light recycling" increases the power emitted by the phosphor, and the power passing through the aperture, compared to power emitted by the phosphor of an "ordinary" lamp.

*As shown in the figure on the previous page, the aperture subtends angular range $d\theta = \Delta\phi/2$ about a segment $ds$ of the phosphor at azimuth $\phi$. The probability that a photon emitted by the segment $ds$ immediately escapes through the aperture is $(d\theta\,\cos\theta)/2 = (\Delta\phi\cos\theta)/4$ for a Lambert-law radiator. Averaging this probability over the range $2\pi$ of angle $\phi$ corresponds to averaging over the range $\pi$ of angle $\theta = \phi/2$, yielding $\Delta\phi/2\pi$ as the probability that a photon emitted somewhere on the phosphor immediately escapes. Hence, a typical photon escapes through the aperture only after $N = 2\pi/\Delta\phi$ cycles of absorption and re-emission.*

Recalling eq. (4), we note that the angular distribution of the power that emerges through the small aperture is,

$$\frac{dP_{\text{aperture}}}{d\theta} = \frac{P_0\cos\theta}{2} \qquad (-\pi/2 < \theta < \pi/2). \tag{7}$$

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[7]The flux of radiation inside a cavity whose walls obey Lambert's law (3) is isotropic.

If the light from the aperture lamp is intercepted by a (half) cylindrical screen of radius $r \gg a$, the light intensity at the screen is

$$I_{\text{aperture}}(r, \theta) = \begin{cases} P_0 \cos\theta/2r & (-\pi/2 < \theta < \pi/2), \\ 0 & (\pi/2 < \theta < 3\pi/2). \end{cases} \tag{8}$$

The intensity (8) is $\pi$ times that of eq. (1) for an "ordinary" lamp of the same luminous power incident on a cylindrical screen of radius $r$ centered on the lamp (rather than on the aperture). *This result was the original motivation for the development of aperture lamps.*

## 2.2 Brightness

Another quantity of interest in optical systems is brightness. If a lamp is used to illuminate a sheet of paper, we colloquially speak of the brightness of the paper, which is a diffuse $\approx$ Lambertian) reflector of the light incident upon it. As such, the brightness of a region of the paper is proportional to the power incident upon it, *i.e.*,

$$\text{Brightness}_{\text{paper}} \propto \frac{dP_{\text{incident}}}{d\text{Area}}. \tag{9}$$

If the light source has a fixed power output, the brightness of the paper can be increased by concentrating/focusing the light on the paper by a lens or a reflector, *etc.*

The brightness of the paper can be increased because the paper is a diffuse reflector of light, rather than a mirror.

It is customary (although perhaps confusing) in optics to speak also of the brightness of a "beam" of light, where a beam is a flow of optical energy (light) that has an axial symmetry, limited extent transverse to the optic axis, and all axial momentum components with the same sign. A beam can have can have arbitrary time dependence, although in optics (and in this note) it is generally assumed that beams are steady in time. The brightness $B$ (or radiance $L$) of a beam two or three dimensions is defined such that the brightness of a Lambertian surface is independent of $\theta$,[8]

$$B_{2-\text{d}} = \frac{1}{\cos\theta} \frac{d^2 P}{ds\, d\theta}, \qquad \text{and} \qquad B_{3-\text{d}} = \frac{1}{\cos\theta} \frac{d^2 P}{d\text{Area}\, d\Omega}, \tag{10}$$

where $P$ is power, the arc segment $ds$ and the area element $d\text{Area}$ are perpendicular to the optic axis, and angle $\theta$ is with respect to that axis.[9] The brightness $B$ of eq. (10) is in general a function of angle $\theta$; often THE brightness of the beam is taken to be the value of $B$ at $\theta = 0$ with respect to the optic axis.

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[8]Radiation inside a "black" cavity is isotropic, such that the power incident on the cavity surface at angle $\theta$ to its inward normal varies as $\cos\theta$.

[9]Following eq. (10), we can define the 3-d brightness of light reflected from Lambert's-law paper as

$$B_{\text{paper}} = \frac{1}{\cos\theta} \frac{d^2 P_{\text{reflected}}}{d\text{Area}\, d\Omega} = \frac{1}{\cos\theta} \frac{dP_{\text{reflected}} \cos\theta}{\pi\, d\text{Area}} = \frac{dP_{\text{reflected}}}{\pi\, d\text{Area}} = \frac{dP_{\text{incident}}}{\pi\, d\text{Area}}. \tag{11}$$

Although $B_{\text{paper}}$ can be increased by focusing the incident light on the paper, the "brightness theorem" discussed in sec. 2.3 tells us that $B_{\text{paper}} < B_{\text{incident}}$.

According to definition (10) the brightness of the phosphor surface inside the aperture lamp is, recalling eqs. (3) and (6),

$$B_{\text{phosphor}} = \frac{1}{\cos\theta} \frac{d^2 P_{\text{phosphor}}}{ds\,d\theta} = \frac{P_0}{2a\Delta\phi} \qquad \text{(aperture lamp)}. \qquad (12)$$

For comparison, the outward brightness of the phosphor at radius $a$ of an "ordinary" lamp is, using eq. (1),

$$B_{\text{ordinary}} = B_{\text{phosphor}} = \frac{1}{\cos\theta} \frac{dP_{\text{phosphor}}}{ds} \frac{\cos\theta}{2} = \frac{P_0}{4\pi a}. \qquad (13)$$

The light emerging from the lamp is spread over the full azimuthal range $2\pi$ in case of an "ordinary" lamp, and range $\pi$ in case of an aperture lamp. In neither case is this behavior much like that of a "beam", but we can define an optic axis for an aperture lamp to be at any angle $\theta$ about the center of the aperture, measured with respect to the ray from the center of the lamp through the aperture. The brightness $B_{\text{aperture}}(\theta)$ on the surface of the aperture of size $ds = a\Delta\phi$ follows from eq. (8) as

$$B_{\text{aperture}}(\theta) = \frac{1}{\cos\theta} \frac{d^2 P_{\text{aperture}}}{ds\,d\theta} = \frac{1}{a\Delta\phi\cos\theta} \frac{dP_{\text{aperture}}}{d\theta} = \frac{P_0}{2a\Delta\phi} = B_{\text{phosphor}}. \qquad (14)$$

*The brightness (14) of the aperture of an aperture lamp is greater than the surface brightness (13) of "ordinary" lamp with the same power output $P_0$ by the factor $N = 2\pi/\Delta\phi$, i.e., $B_{\text{aperture}} = NB_{\text{ordinary}}$. This is because a photon emitted by the phosphor immediately leaves an "ordinary" lamp, but is absorbed and re-emitted by the phosphor an average of $N$ times before it finally escapes through the aperture of an aperture lamp. In language suggested by [4], the light is "recycled" (stored) inside the aperture lamp for $N$ "lifetimes" of $2\sqrt{2}a/\pi c = 0.9a/c = $ average transit time of a photon across the lamp of radius $a$ if emitted isotropically from the surface, where $c$ is the speed of light.[10]*

A segment $ds'$ on a distant cylindrical surface of radius $r \gg a$ at angle $\theta$ to the normal to the aperture subtends angle $d\theta = ds'/r$ with respect to the aperture, intercepts power $dP = (ds'/r)dP_{\text{aperture}}/d\theta = (ds'/r)(1/2)P_0\cos\theta$. The angular range $d\theta'$ of this radiation with respect to the segment is $d\theta' = a\Delta\phi\cos\theta/r$ at angle $\theta' = 0$ with respect to the normal to the segment $ds'$, so the brightness at the distance segment is,[11]

$$B(r,\theta) = \frac{1}{\cos\theta'} \frac{d^2 P}{ds'\,d\theta'} = \frac{(ds'/r)(1/2)P_0\cos\theta}{ds'(a\Delta\phi\cos\theta/r)} = \frac{P_0}{2a\Delta\phi} = B_{\text{phosphor}} = B_{\text{aperture}}. \qquad (15)$$

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[10]A common example of "light recycling" is a pair of "ordinary" fluorescent lamps in close proximity. Light from one lamp that is incident on the other results in absorption and re-emission by the phosphor, which increases the brightness of both lamps, particularly in the regions of their surfaces closest to one another. See the discussion related to Fig. 2 of [4].

[11]An early statement of this "brightness theorem" was given in [9] (1888).

## 2.3 Étendue

The result (15) illustrates the lore that "brightness" is an invariant in lossless optical systems for which diffraction can be ignored (as assumed in this note). The underlying notion is conservation of energy (or conservation of number of rays, as in the original statement by Lagrange [10]). Namely, if a set of rays emanate from an object of height $h$ within some angle $\theta$ to the optic axis of a (2-d) lens of height $h''$ at distance $r$, and are focused to an image of height $h' = r'h/r$ at distance $r$ from the lens, as shown below, then $h'' = 2r\sin\theta/2 = 2r'\sin\theta'/2$, so $h\theta = h'\theta'$ for small angles.
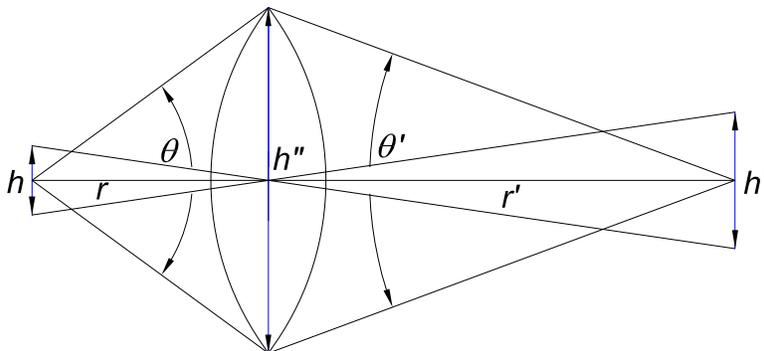


If there is no loss of energy (or rays) in the system the emitted power $P$ equals the received power $P'$ and the object and image brightnesses are related by

$$B = \frac{P}{h\theta} = \frac{P'}{h'\theta'} = B'. \tag{16}$$

An image in an optical system can be bigger than the object, but not brighter. However, this version of the "brightness theorem" holds only for imaging optical systems.

In eq. (16) the numerators and the denominators of the fractions are separately equal. The equality of the numerators is due to conservation of energy, while the equality of the denominators is a geometric result of imaging optics. The latter is noteworthy, and has led to the notion of conservation of étendue,[12] $\epsilon$, where in a medium of index of refraction $n$ the étendue (geometric extent) is

$$d^2\epsilon_{2-\mathrm{d}} = n\cos\theta \, ds \, d\theta, \qquad d^2\epsilon_{3-\mathrm{d}} = n^2\cos\theta \, d\mathrm{Area} \, d\Omega, \tag{17}$$

and brightness can now be written as

$$B_{2-\mathrm{d}} = \frac{1}{\cos\theta}\frac{d^2P}{ds \, d\theta} = \frac{d^2P}{d^2\epsilon_{2-\mathrm{d}}}, \qquad \text{and} \qquad B_{3-\mathrm{d}} = \frac{1}{\cos\theta}\frac{d^2P}{d\mathrm{Area} \, d\Omega} = \frac{d^2P}{d^2\epsilon_{3-\mathrm{d}}}. \tag{18}$$

A geometrical-optics argument for the conservation of étendue in 2-d systems can be given using the figure on the next page. The premise is that all the power $d^2P$ (*i.e.*, all the

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[12]The more complete French technical term is l'étendue géométrique du faisceau, the geometric extent of beams. The French word étendue (area, extent, range) may have first appeared in English in the technical context of eq. (17) in the English abstract of the French paper [11], and next in sec. E of the review [12] which also may be the first application of Liouville's theorem to optics. Use of the name étendue was discouraged in [13] and is not common except in the literature of light concentrators.

rays) emitted by (or passing through) surface $ds$ within some narrow angular range $d\theta$ about angle $\theta$ to the normal to $ds$ are intercepted by (or pass through) surface $ds'$. To illustrate the role of the index of refraction in the definition of étendue, we suppose that surface $ds'$ lies on the interface between media of indices $n_1$ and $n_1$, where surface $ds$ is within the medium of index $n_1$.
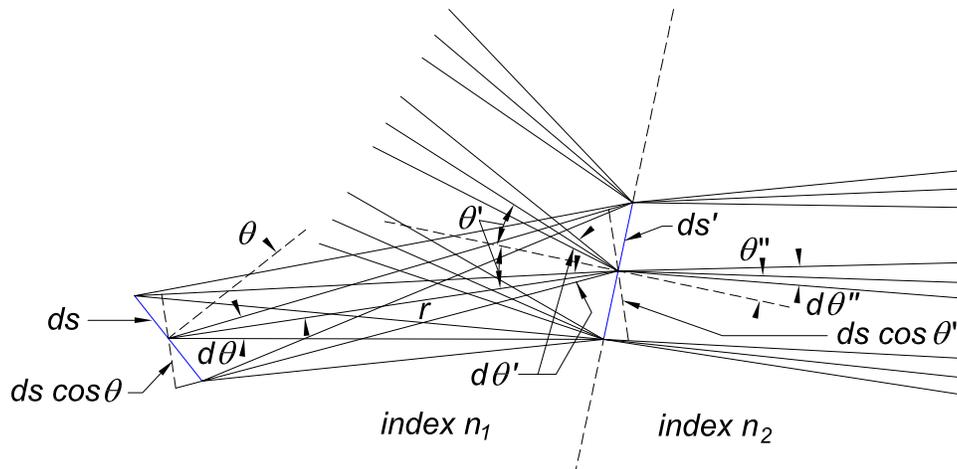


We see that the angular range $d\theta$ of the power emitted by surface $ds$ is given in terms of properties of the intercepting surface $ds'$ as

$$d\theta = \frac{ds' \cos \theta'}{r} ,$$  (19)

where $r$ is the distance between the midpoints of the two surfaces, and that the angular range $d\theta'$ of this power about angle $\theta'$ with respect to the normal to the intercepting surface $ds'$ is (ignoring reflections at the surfaces of the lens) given by

$$d\theta' = \frac{ds \cos \theta}{r} .$$  (20)

Hence, the brightness of this power at the two surfaces (in the same medium) is related by

$$B = \frac{1}{\cos \theta} \frac{d^2 P}{ds\, d\theta} = \frac{r\, d^2 P}{\cos \theta\, ds\, ds' \cos \theta'} = \frac{1}{\cos \theta'} \frac{d^2 P}{ds'\, d\theta'} = B',$$  (21)

and the étendues are the same,

$$d^2 \epsilon = n_1 \cos \theta\, ds\, d\theta = n_1 \cos \theta'\, ds'\, d\theta' = d^2 \epsilon'.$$  (22)

It is also clear that a (lossless) mirror reflection does not change the étendue or the brightness,

$$d^2 \epsilon'_{\text{reflected}} = d^2 \epsilon', \qquad \text{and} \qquad B_{\text{in}} = \frac{d^P_{\text{in}}}{d^2 \epsilon'} = \frac{d^P_{\text{reflected}}}{d^2 \epsilon'_{\text{reflected}}} = B_{\text{mirror reflection}}.$$  (23)

Furthermore, as the power passes across the interface between the two media, the rays obey Snell's law, $n_1 \sin \theta' = n_2 \sin \theta''$, so that the small angular ranges $d\theta'$ and $d\theta''$ are related by

$$n_1 \cos \theta'\, d\theta' = n_2 \cos \theta''\, d\theta''.$$  (24)

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Since $ds' = ds''$ for the surface element on the interface, we have that

$$d^2\epsilon'' = n_2 \cos\theta'' \, ds'' \, d\theta'' = n_1 \cos\theta' \, ds' \, d\theta' = d^2\epsilon' = d^2\epsilon. \tag{25}$$

The power incident on the interface is partly reflected and partly transmitted, $d^2 P_{\text{in}} = d^2 P_{\text{reflected}} + d^2 P_{\text{transmmited}}$, so brightness is conserved at the interface in the sense that

$$B_{\text{in}} = \frac{d^2 P_{\text{in}}}{d^2\epsilon} = \frac{d^2 P_{\text{in}}}{d^2\epsilon'} = \frac{d^2 P_{\text{reflected}}}{d^2\epsilon'_{\text{reflected}}} = \frac{d^2 P_{\text{transmitted}}}{d^2\epsilon''} = B_{\text{reflected}} + B_{\text{transmitted}}. \tag{26}$$

### 2.3.1  Can a Reflector Increase Optical Brightness?

Lenses and reflectors are often used with light sources to increase the amount of light delivered into some distant area. If that area is a diffuse reflector, its optical brightness is thereby increased. However, the optical brightness of the beam of light incident on that area is not increased by the lens, as argued above. When a "back" reflector is used and the reflected light passes around source, light is added to the "forward" beam, but along different rays than those emitted directly "forward" from the lamp. This extends that range of angles for which the brightness of the source is defined (and increases the amount of light delivered into the distant area), but it does not increase the brightness of the direct "forward" beam.

In addition, there is the possibility that a ray emitted "backwards" from the source is reflected back onto the source and passes through it, adding to the light from the source in the "forward" direction. This process would increase the optical brightness of the "forward" beam of light from the source. Most light sources are not transparent to their own radiation, so that this mechanism of enhancement of optical brightness is little realized in practice. However, light that is reflected back onto the source and absorbed by it can be reradiated, with some of the reradiation emitted into the "forward" beam, thereby enhancing its optical brightness. This process is the conceptual precursor to that of an aperture lamp.

### 2.3.2  Étendue and an Aperture Lamp

Returning to the example of an aperture lamp of radius $a$ and small angular aperture $\Delta\phi$, all in media with index $n = 1$, we consider a segment $ds$ on the phosphor at distance $b$ from the aperture $ds'$, and the segment $ds'''$ on a distant screen at $r \gg a$ that intercepts the light emitted by segment $ds$ which passes through the aperture.



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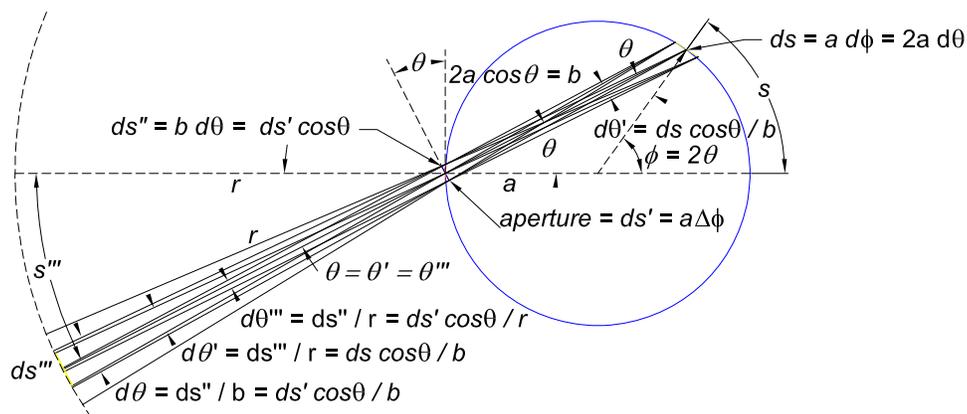
The angular extents $d\theta$, $d\theta'$ and $d\theta'''$ of this light are related by

$$d\theta = \frac{ds' \cos\theta}{b}, \qquad d\theta' = \frac{ds \cos\theta}{b} = \frac{ds'''}{r}, \qquad \text{and} \qquad d\theta''' = \frac{ds' \cos\theta}{r}, \qquad (27)$$

from which we learn that $ds''' = r\, ds\, \cos\theta/b$ (rather than $ds''' = r\, ds/b$ as might naïvely be expected if we think of the aperture as a pinhole lens[13]). Hence, the étendues are related by

$$d^2\epsilon = \cos\theta\, ds\, d\theta = \frac{ds\, ds' \cos^2\theta}{b} = \cos\theta'\, ds'\, d\theta' = d^2\epsilon' = \frac{ds'''\, ds' \cos\theta}{r} = ds'''\, d\theta''' = d^2\epsilon''', (28)$$

noting that $\theta = \theta' = \theta'''$ and that the light incident on element $ds'''$ is at zero angle to its normal.

We can also consider the total étendue $\epsilon$ of the phosphor, $\epsilon'$ of the aperture, and $\epsilon'''$ of the distant screen,

$$\epsilon = \int\int d^2\epsilon = \int_{-\pi a}^{\pi a} ds \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = 2\pi a \int_{-1}^{1} d\sin\theta = 4\pi a, \qquad (29)$$

$$\epsilon' = \int\int d^2\epsilon' = \int_{0}^{a\Delta\phi} ds' \int_{-\pi/2}^{\pi/2} \cos\theta'\, d\theta' = a\Delta\phi \int_{-1}^{1} d\sin\theta = 2a\Delta\phi, \qquad (30)$$

$$\epsilon''' = \int\int d^2\epsilon''' = \int_{-\pi r/2}^{\pi r/2} ds''' \int_{0}^{a\Delta\phi\cos\theta'''/r} d\theta''' = a\Delta\phi \int_{-\pi r/2}^{\pi r/2} \frac{ds'''}{r} \cos\frac{s'''}{r} = 2a\Delta\phi, (31)$$

where in eq. (31) we noted that $\theta''' = s'''/r$. We should also note that the total étendue $\epsilon$ of the phosphor is much larger than the étendue $\epsilon_{\text{output}}$ of the light emitted by the phosphor which passes through the aperture,

$$\begin{aligned}
\epsilon_{\text{output}} &= \int\int_{\text{output}} d^2\epsilon = \int_{-\pi a}^{\pi a} ds \int_{\theta}^{\theta+a\Delta\phi\cos\theta/b} \cos\theta\, d\theta = a\Delta\phi \int_{-\pi a}^{\pi a} \frac{ds}{2a} \cos\frac{s}{2a} = 2a\Delta\phi \\
&= \epsilon' = \epsilon''',
\end{aligned} \qquad (32)$$

recalling that the aperture subtends angle $a\Delta\phi\cos\theta/b = (1/2)\Delta\phi$ and that $\theta = \phi/2 = s/2a$.

The brightness of the aperture lamp is $B = NP_0/\epsilon = P_0/\epsilon_{\text{output}} = P_0/\epsilon' = P_0/\epsilon''' = P_0/2a\Delta\phi$, as found previously in eq. (15).

Hence, **the aperture lamp obeys the brightness theorem** in the sense that the brightness and étendue of light at a distance cylinder is the same as the brightness and étendue of the light emitted by phosphor into the aperture. However, **the brightness $P_0/2a\Delta\phi$ of the phosphor inside an aperture lamp is much greater than the brightness $P_0/4\pi a$ of the phosphor inside an "ordinary" lamp of the same luminous power output**. Hence, the interest in the present example is that while brightness is invariant during optical transport, the initial brightness can be increased by the "trick" of "recycling" light back onto the initial emitting surface such that each emitted photon contributes many times to the photometric measure of brightness.

*There are (at least) two ways to think about the beneficial effect of "light recycling".*

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[13]When considering the geometry of finite surface elements $ds$ and $ds'$, as in the figure above, some rays emanating from $ds$ which pass through the aperture $ds'$ are outside the element $ds'''$, while this is not the case in the limit of "infinitesimal" elements.

1. If we focus on the phosphor inside the aperture lamp, we note in eq. (6) that the effect of "light recycling" is to increase the power emerging from the phosphor by the factor $N$ = number of times each photon is "recycled" before it emerges through the aperture of the lamp. However, the "recycling" has no effect on the étendue into which light from the phosphor is emitted, so the brightness of the phosphor is $N$ times large in an aperture lamp than in an "ordinary lamp". Then, the brightness at the aperture, and on a distant screen are the same as that of the phosphor, in accordance with the "brightness theorem", and these are all $N$ times larger than the brightness at the surface of an "ordinary" lamp, and at a screen distant therefrom.

   If the phosphor did not "recycle" the light that strikes it, and simply absorbed it, the output power of the lamp would be reduced by a factor $\Delta\phi/2\pi$, and the output brightness would be $(P_0 \Delta\phi/2\pi)/\epsilon_{\text{output}} = P_0/4\pi a$, which is the same as that of an "ordinary" lamp.

2. If instead we focus on the aperture of the aperture lamp, we note that all of the output power $P_0$ of the lamp flows through the aperture, and into étendue $2a\Delta\phi$. In contrast, the power emerging from an "ordinary" lamp flows into étendue $4\pi a$. From this perspective, the brightness $P_0/2a\Delta\phi$ of the aperture lamp is greater than the brightness $P_0/4\pi a$ of an "ordinary" lamp by the factor $N = 2\pi/\Delta\phi$ because the photons are "recycled" inside the lamp an average of $N$ times until they finally emerge through the aperture whose azimuthal extent is only $\Delta\phi/2\pi$ of the total possible.

   In this view, the étendue of the photons which emerge from the lamp is smaller by the factor $1/N = \Delta\phi/2\pi$ compared to that of all photons emitted by the phosphor, and to that of photons emerging from an "ordinary" lamp. This is not a violation of "conservation of étendue", as the comparison involves different "beams", rather than different places in a single "beam". Indeed, eq. (32) shows that the étendue of the output "beam" is conserved from place to place in an aperture lamp.

For additional insight as to how/why the brightness of the phosphor can be increased by "light recycling", we consider the relation between étendue and emittance in the following section.

### 2.3.3  Étendue, Hamilton, Liouville, and Emittance

In the 2-d étendue, $d^2\epsilon = \cos\theta\, ds\, d\theta$, we note that $\cos\theta\, d\theta = d\sin\theta$, so if the element $ds$ lies along, say, the $x$-axis, the étendue can be written as

$$d^2\epsilon = \frac{dx\, dp_x}{p} , \tag{33}$$

where $p_x = p\sin\theta$, $p = \hbar k = 2\pi\hbar/\lambda$ is the momentum of a photon of the light of wavelength $\lambda$, and $\hbar = h/2\pi$ with $h$ being Planck's constant. Similarly, the power $d^2P$ can be written as

$$d^2P = \hbar\omega\frac{dN}{dt} , \tag{34}$$

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where $\omega = 2\pi f$ is the angular frequency of light, and $dN/dt$ is the number of photons that cross the element $dx$ each second. Then, the brightness of the light beam can be written as

$$B = \frac{d^2 P}{d^2 \epsilon} = \frac{\omega}{k} \frac{dN/dt}{dx \, dp_x} = \frac{v_{\text{phase}} \, dN/dt}{dx \, dp_x} \,, \tag{35}$$

where the phase velocity is $v_{\text{phase}} = \omega/k = c/n$, $c$ is the speed of light in vacuum and $n(\omega)$ is the index of refraction.

The quantity $dx \, dp_x$ is an element of (transverse) phase space, where the latter is the space $(x, z, p_x, p_z)$ that is considered in 2-d Hamiltonian dynamics [14] (with $z$ being considered here as the longitudinal coordinate). Thus, the optical brightness $B$ is proportional to the density of photons in transverse phase space $(x, p_x)$.

A famous theorem in classical dynamics is that the extent of a phase space occupied by a system is invariant under time evolution for systems that can be described by a Hamiltonian function (and the time evolution described by a "canonical transformation".[14,15] This important result is ascribed to Liouville [18] (who preceded Hamilton and never actually proved "Liouville's theorem", as discussed in [19]).

The relation between classical dynamics and geometrical (ray) optics is not self evident.[16] No such connection was made by Hamilton.[17] The first connection between Liouville's theorem and beams of particles seems to have been made by Swann in 1933 [22] when considering electrons in the Earth's ionosphere. The first discussion of Liouville's theorem in relation to optical systems may be by Joyce in 1974 [12] (which also seems to be the first use of the term étendue in an English-language paper).[18]

System with losses and/or diffuse reflectors cannot be described by a Hamiltonian function, so care is required in applying Liouville's theorem, and conservation of étendue, to them. Once light has left the surface of the phosphor (or other diffuse reflector), the light obeys Hamiltonian dynamics (if losses can be ignored) and its theorems apply. However,

[14]A succinct summary of this lore is given in sec. 46 of [15]. Not only is an element $dx \, dp_x \, dy \, dp_y \, dz \, dp_z$ in 6d phase space conserved by canonical transformations, but also elements such as $dx \, dp_x$ and $dx \, dp_x \, dy \, dp_y$ in 2-d and 4d subspaces are conserved.

[15]A misunderstanding in, for example, Appendix A.3 of [16] is that Liouville's theorem is "statistical", and therefore somehow "fundamentally different" from conservation of étendue in optics. However, "proofs" of Liouville's theorem (as in [15]) discuss the evolution of volumes in phase space; application to statistical mechanics can be made thereafter, but the theorem itself is purely geometrical.

[16]It is sometimes claimed that the connection was made in the 1944 lectures by Luneberg [20], which include an extensive discussion of Hamiltonian optics [21], but which never mention the theorem of Liouville, and which fail to note that the Jacobian of eq. (20.11) is unity according to that theorem.

[17]Although Hamilton is associated with sophisticated developments in geometrical (ray) optics ($\approx 1840$), following the spirit of Fermat and Newton that light is a particle phenomenon, he never made a connection between his theories of optics and his theories of the dynamics of massive systems. A brief review of this connection is given, for example, in sec. 2.1 of [17].

[18]The first mention of phase space in relation to an optical system may be in footnote 3 of the 1952 paper by Garwin [23, 24] on collection of Čerenkov light in "adiabatically" tapered plastic light guides, although the argument there is "thermodynamic" rather than in the spirit of Hamilton/Liouville. The use of Hamiltonian methods in light collection problems was advocated by Winston in 1970 [25], following earlier suggestions by Luneberg [20]; however these works emphasized conservation of angular momentum rather than the invariance of density in phase space. The application of Liouville's theorem to optical was perhaps taken for granted only after 1978 [26].

these theorems do not apply to the process of diffuse reflection (absorption from one angle and emission at a random angle).

We noted at the beginning of sec. 2.2 that the brightness of a diffusely reflecting surface is not simply related to the brightness of an optical beam incident upon it. In the case of an aperture lamp, we have seen in eq. (15) that the brightness of the light at large distance $r \gg a$ is the same as that at the aperture, and also the same as that of the phosphor,

$$B(r, \theta) = B_{\text{aperture}} = B_{\text{phosphor}}. \tag{36}$$

However, the phosphor itself does not obey Hamiltonian dynamics, so it is not required that the brightness of the phosphor in the aperture lamp be the same as that of an "ordinary" lamp of the same luminous power output. Of course, Liouville's theorem and the related conservation of étendue give no insight as to how brightness can be enhanced in situations to which they don't apply.

In discussions of charged-particle beams the extent of particles in phase space has come to be called emittance.[19] The étendue of geometric optics is the equivalent of transverse emittance in particle-beam dynamics.[20]

The brightness enhancement of the phosphor of an aperture lamp, compared to that of an "ordinary" lamp of the same luminous power, is due to an increase (via "light recycling") in the effective power emitted by the phosphor, rather than due to any reduction of the étendue/emittance of the light "beam" after it has been emitted.[21,22]

## 2.4   Aperture Lamp Variants

We now consider several variants of aperture lamps.

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[19]This concept seems to have been invented in 1952 by Sigurgeirsson [27], who called it "admittance" in his eq. (22).

[20]Particle beams are typically pulsed, so longitudinal emittance (in either the $(z, p_z)$ or $(t, E)$ subspaces) is of relevance there. Optical beams are often time-independent, for which longitudinal étendue is not well defined. In the classic phase space $(x, p_x, y, p_y, z, p_z)$, the independent variable is time $t$. When considering beams it is often more convenient to consider them at fixed $z$, rather than fixed $t$, which requires a Hamiltonian formalism with $z$ as the independent variable, and phase space $(x, p_x, y, p_y, t, E)$, as first considered in Appendix B of [28]. See also [29].

[21]This contrasts with schemes for brightness enhancement of particle beams which typically involve reduction ("cooling") of the emittance/phase space of the beam, while (ideally) keeping the beam power constant. See, for example, the 1984 Nobel lecture by Van der Meer [30]. The first proposal for particle-beam cooling was by O'Neill in 1956 [31] via a technique now called "ionization cooling".

[22]"Light recycling" is somewhat equivalent to the use of particle beams in circular "storage rings" whereby $m$ "bunches" of $n$ particles pass an "interaction point" $f$ times per second, and the "luminosity" for a pair of bunches in two intersecting storage rings is $\mathcal{L} = fmn^2/\text{Area}$, such that the particle interaction rate is $R = \mathcal{L}\sigma$, where $\sigma$ is the interaction cross section. The papers by Kerst [32, 33] that propose this concept credit Wigner with bringing the importance of Liouville's theorem to the attention of the particle-beam community. The bunches in the storage ring are often built up by injection on successive "turns", where each successive bunch is placed into a slightly different region of phase space by a process called "phase-space painting". Thus, the luminosity is enhanced by the "recycling" of the particles each "turn" rather than by a decrease in the area they occupy in transverse phase space.

### 2.4.1 Magic Lantern with Two Light Sources

A magic lantern is a simple optical projector (first described by Huygens (1671) using the left figure below [34]), which typically uses a spherical mirror to focus a light source onto a small semitransparent object, leading to a larger image on a distant screen. A variant [35] from 1864 uses two light sources to illuminate a piece of paper, whose reflected light is projected on a distant screen, is shown on the right below. The diffuse reflection of the paper combines light from both sources into the output beam, which could not be accomplished in the original scheme using mirrors. Thus, the brightness is enhanced by a non-Hamiltonian process which is a precursor to the "light recycling" of an aperture lamp.
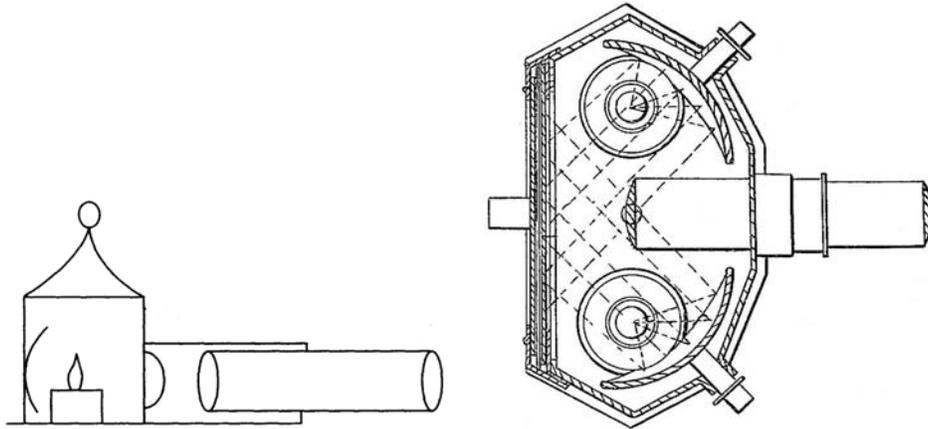


### 2.4.2 Phosphor Over Only Part of the Azimuth

Suppose the phosphor extends only over an azimuthal range $\Delta\phi_p$, with the remainder of the azimuth occupied by either the small aperture of azimuthal extent $\Delta\phi$, or by a perfect reflector, as below.
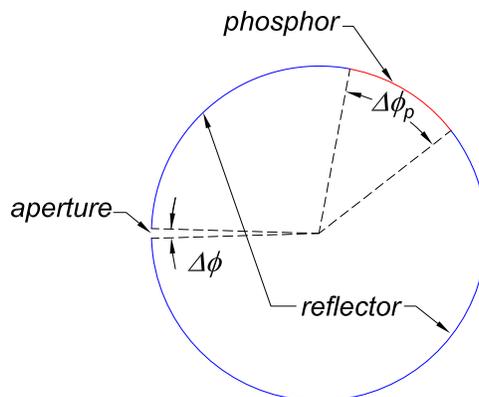


If the phosphor re-emits all the light that strikes it, then all light emitted by the phosphor eventually escapes through the aperture. Also, the reflector does not change the angular distribution of light incident upon it, so the angular distribution of light emerging through the aperture is the same as if the phosphor occupied all of the azimuth (except that of the aperture). Hence, the output intensity, brightness and étendue of this variant are identical to those of the aperture lamp previously described, independent of the azimuth and size of

the phosphor, if the power $P_0$ that excites the phosphor is the same.

In particular, the angular extent $\Delta\phi_p$ of the phosphor could be made very small, and the effect of "light recycling" would be unchanged in principle. This raises the question as to whether the diffuse relection of the phosphor is actually critical to the process of brightness enhancement.

### 2.4.3   The Phosphor Does Not Re-emit its Own Radiation

We consider another variant in which the phosphor absorbs light that strikes it, but does not re-emit it. In particular, suppose the phosphor is directly opposite the aperture, and has the same azimuthal extent, $\Delta\phi_p = \Delta\phi$, as shown below.
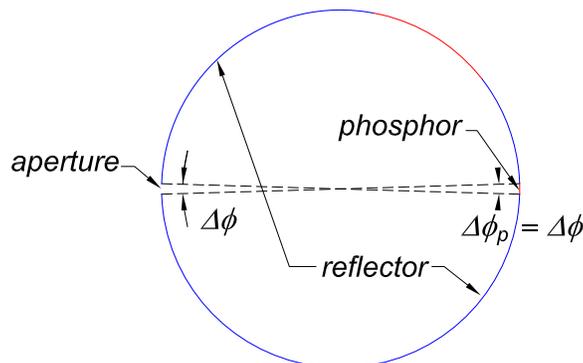


In this case, no light can be reflected into the narrow "beam" of light that emerges from the phosphor and passes directly through the aperture. There is no enhancement of the brightness of this beam (whose brightness is just that of the phosphor itself). Reflected light does emerge through the aperture outside this "beam", into azimuths where there would be no light at all without the reflector. We can say that the reflector has "enhanced" the brightness (up from zero brightness otherwise) outside the direct "beam" from the phosphor, but there is no enhancement of the brightness of the direct "beam".

We conclude that diffuse reflection is the key to "light recycling" no matter how small the azimuthal extent of the phosphor.

### 2.4.4   The Phosphor is a Perfect Reflector of its Own Radiation

In classical optics we think of a mirror as a passive device that reflects light (specularly), but which does not emit it. If a "phosphor" exists that can emit light at some frequency, but which reflects incident light specularly at that frequency, an aperture lamp made with such a phosphor (over all azimuth except that of the aperture) would exhibit brightness enhancement via "light recycling".

After the initial emission of a photon by such a phosphor, the optical transport is "Hamiltonian", and this example shows that brightness enhancement via "light recycling" is possible, in principle, in a system governed by Hamiltonian optics.

*It is unclear that the desired "phosphor" exists. Fluorescent materials exhibit a "Stokes' shift" [36] between frequencies of emission and absorption, but these materials reflect diffusely rather than specularly. So-called* photonic band gap *materials exhibit specular reflec-*

*tion at certain frequencies, but this phenomenon is associated with the absence of (sponta-neous) emission at these frequencies [37].*

### 2.4.5 Brightness Enhancement of Polarized Light

If only one polarization of the output light is desired, a polarizer could be put at the aperture of the lamp. Ordinarily, this would eliminate half of the otherwise unpolarized light. However, if a so-called reflecting polarizer[23] is used, light of the undesired polarization is reflected back into the lamp, and becomes unpolarized (*i.e.*, randomly polarized) again upon diffuse reflection by the phosphor. When this light emerges through the aperture, half of it adds to the brightness of the desired polarization, and half is sent back into the lamp, *etc.* If there are no losses in this process, eventually all of the light emerges with the desired polarization, whose brightness is then twice that of the output light of the desired polarization in the absence of the polarizer.[24,25,26]

### 2.4.6 Brightness Enhancement in Two Adjacent "Ordinary" Lamps

Another interesting variant of "light recycling" occurs when two "ordinary" phosphor lamps are adjacent to one another. The light from one lamp that is intercepted by the other is absorbed and re-emitted, adding to the brightness of the latter lamp, and *vice versa*.[27]

### 2.4.7 The Aperture Lamp is Filled with a Medium of Index $n > 1$

If the space inside the aperture lamp is filled with a medium of index of refraction $n > 1$, light emerges through the aperture only if its angle inside the lamp obeys $|\sin\theta| < 1/n$, according to Snell's law. The computation (32) of the étendue $\epsilon_{\text{output}}$ of the output light when still inside the lamp has a factor $n$ in the integrand, while the limits of integration in $s$ are smaller by a factor $\sin^{-1}(1/n)$, such that the result is unchanged. Also, computations (30)-(31) of the étendues $\epsilon'$ and $\epsilon''''$ are unchanged. The brightness of the aperture lamp is again $B = P_0/\epsilon_{\text{output}} = P_0/2a\Delta\phi = P_0/\epsilon' = P_0/\epsilon'''$.[28]

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[23]Planar reflecting polarizers, based on nanofabrication, are a relatively recent development. See, for example, [38].

[24]This procedure was suggested in Fig. 9 of [5]. In Hamiltonian optics it would not be possible to convert unpolarized light into light of a single polarization, with all light in the same étendue/emittance as the unpolarized light. This conversion by "light recycling" off a diffusely reflective phosphor is an additional advantage of this technique.

[25]The conversion of unpolarized light by the combination of "measurement" by a reflective polarizer and randomization by diffuse reflection is an example of the quantum watchdog effect [39], which is a variant of the quantum Zeno effect [40].

[26]If the phosphor in the lamp were the hypothetical material of sec. 2.4.3, the photons reflected back into the lamp would remain polarized and there would be no enhancement of the brightness of the light that passes through the polarizer (beyond the brightness enhancement considered in sec. 2.2).

[27]This effect was suggested in Fig. 2 of [4]. It is unrelated to the "quantum" effect discussed in [42] which arises when the spacing is less than an optical wavelength.

[28]A recent eprint [41] seems unaware that the brightness enhancement in their experiment is due to "light recycling", and that the use of a medium of index $n$ inside the cavity has no effect on the brightness.

## 2.5  Spectrum and Temperature of an Aperture Lamp

### 2.5.1  The Phosphor Has a Line Spectrum

We have tacitly assumed in previous sections that the phosphor of the aperture lamp emits light of a single frequency. This is not the case for many fluorescent lamps, but it is a good approximation for a blacklight with europium-doped strontium fluoroborate ($SrB_4O_7F{:}Eu^{2+}$) [43] as the phosphor, whose absorption and emission spectra are shown below.
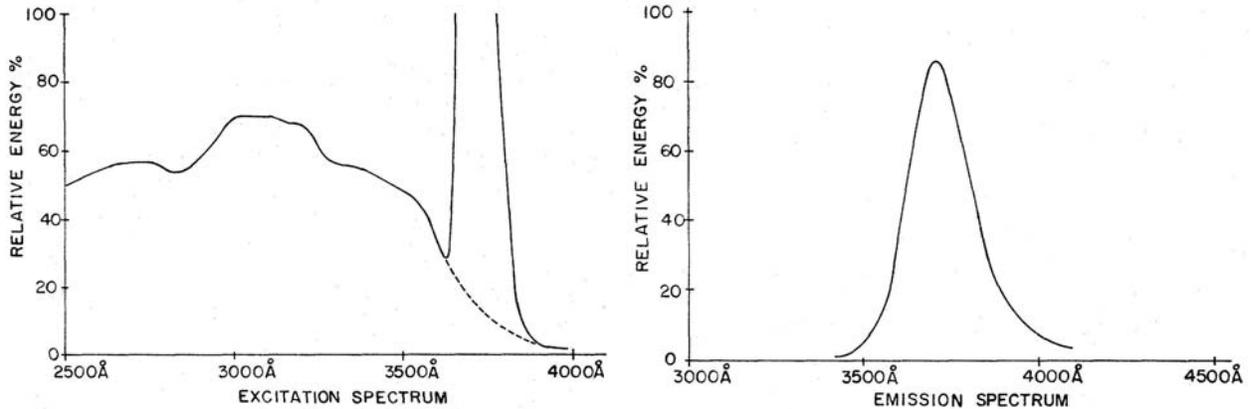


The spectral width of the phosphor is 20 nm, centered on 370 nm. A blackbody with peak emission at this wavelength would have temperature about 7700K, applying Wien's law [44], $T[\text{K}] = 2.9 \times 10^6/\lambda[\text{nm}]$, to a Planckian spectrum [45]. One often speaks of the Wien-law temperature as the temperature of the light in the narrow spectral line. However, the glass surface[29] of the lamp is essentially at room temperature; there is almost no coupling between the process of emission and absorption by the phosphor of the 370-nm light and the thermal energy of the phosphor.[30]

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[29]The lamp should be made of a glass that transmits light near 370 nm, such as so-called Wood's glass. A contemporary version of Wood's glass is the Baader U-filter,
http://www.baader-planetarium.de/sektion/s44/s44.htm
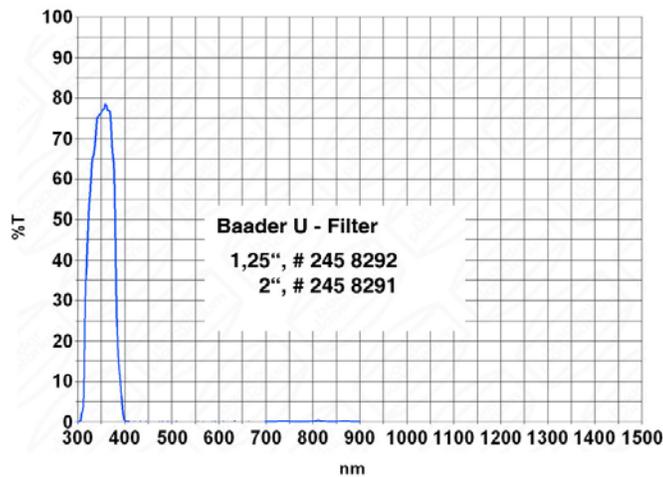


[30]The Wood's glass covering the aperture absorbs about 25% of the light's energy. If this energy is reradiated with a blackbody spectrum, then, say, a 20-W lamp with surface area 0.05 m$^2$ (1$''$ diameter, 24$''$ long) would emit 7 W of blackbody radiation and have a surface temperature of about 225K (if radiating into

### 2.5.2   The Phosphor is a Perfect Blackbody

Another idealization is that the phosphor is a perfect blackbody, with unit absorptivity and emissivity at all frequencies, which is excited to temperature $T$ by the power source.[31] The light output of the aperture lamp has a Planck spectrum of temperature $T$.

The value of the temperature $T$ for a given power $P_0$ is affected by the "light recycling" in the aperture lamp.[32] The temperature of a blackbody is related to is brightness by the Stefan-Boltzmann law [46, 47], which for a cylindrical body has the form

$$B = \frac{d^2 P}{ds\, d\theta\, \cos\theta} = \frac{\sigma T^4}{2} ,  \tag{37}$$

where $\sigma = 2\pi^5 k^4/15 h^3 c^2 = 5.67 \times 10^{-8}$ J/m²sK⁴ is the Stefan-Boltzmann constant. Since the brightness of an aperture lamp is $2\pi/\Delta\phi$ times that of an "ordinary" lamp of the same output power, the temperature of a blackbody aperture lamp is $\sqrt[4]{2\pi/\Delta\phi}$ times that of an "ordinary" blackbody lamp (and the spectral width is smaller by the factor $\sqrt[4]{\Delta\phi/2\pi}$.), assuming that the glass envelope of the lamp is a perfect thermal insulator. The temperature of an aperture lamp, whose brightness is $B_{\text{aperture}} = P_0/2a\Delta\phi$, is $T_{\text{aperture}} = \sqrt[4]{P_0/a\sigma\Delta\phi}$, compared to $T_{\text{ordinary}} = \sqrt[4]{P_0/2\pi a\sigma}$.

### 2.5.3   The Phosphor has Arbitrary Absorption and Emission Spectra

In general the phosphor has different absorption and emission spectra (as noted by Stokes [36]), which do not satisfy Kirchhoff's idealization ("law") that the absorption and emission coefficients are equal [48]. The notion of an effective temperature associated with each wavelength of the light inside a cavity (such as the interior of an aperture lamp) was advocated by Planck in sec. 93 of [49], which leaves open the question of whether the phosphor itself can be said to have a temperature.

If the phosphor exchanges energy with some other entity (possibly an ideal blackbody) that has a temperature $T$, then in a steady-state situation the phosphor can also be said to

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empty space), according to the Stefan-Boltzmann law [46, 47]. That is, a room-temperature environment would warm the glass, rather than the reverse.

[31]The term "blackbody" was introduced by Kirchhoff [48], for whom "black" meant that incident radiation was neither transmitted nor specularly reflected. However, as stated in the first sentence of Kirchhoff's paper, a blackbody inside a cavity must emit as many "rays" as it absorbs. That is, blackbodies do not appear to be black in view of this emission. (The Sun is a good approximation of a blackbody.) In this note we consider the emission of light by a blackbody phosphor inside an aperture lamp at the same rate as the absorption of light to be a kind of generalized reflection, albeit with a Lambertian angular distribution rather than specular reflection.

[32]In the late 1800's it was realized that a good approximation to an ideal blackbody, needing only modest input power to maintain a constant temperature, is a cavity with light being exchanged between the various regions of its internal surface. "Recycling" of the light within the cavity increases the number density of photons over that which would hold if every generated photon were simply "radiated to infinity". As such, the temperature of the cavity can be much higher than if the same input power were delivered to an object of the same surface area, but with only an external surface. Hence, cavity radiation was the first prominent application of "light recycling". However, "light recycling" is not synonymous with blackbody cavity radiation, as the former process can take place at a single wavelength, as assumed in this note until now.

have temperature $T$. A consequence [50] is that the brightness at some wavelength of the surface of the phosphor at temperature $T$ cannot exceed the surface brightness of an ideal blackbody at that temperature.[33,34]

## 2.6 The Aperture Lamp as an Optical Heat Pump or as a Laser

It is suggestive to think of an aperture lamp as an optical heat pump that converts the input power $P_{\rm in}$ into optical output power $P_0$. In case of an ordinary heat pump, the output power is larger that the input power, in that energy is extract from a cold reservoir and delivered to a hot reservoir. However, in an aperture lamp the phosphor is excited by ultraviolet light of shorter wavelength that the output light, and the energy of the UV photons that does not appear in the optical photons is dissipated as heat in the phosphor; $P_0$ is less than $P_{\rm in}$. That is, an aperture lamp is not well modeled as an optical heat pump (unless the phosphor exhibits anti-Stokes behavior, as mentioned further below).[35]

Another appealing view is that an aperture lamp is a kind of laser in which the ultraviolet light is the pump beam, and the visible light is the output beam. For this, it is useful to introduce yet another effective temperature for a light beam (as perhaps first done by Weinstein [55, 56]), being the ratio of the energy flux in the beam to its entropy flux

$$T_{\rm eff} = \frac{dU/dt \, d{\rm Area}}{dS/dt \, d{\rm Area}} \,, \tag{38}$$

where the energy $U$ and entropy $S$ is that of the light in excess of the energy of light emitted by a blackbody with the temperature of the material of the lamp (called $T_{\rm phosphor}$ below). This effective temperature is different than the "brightness temperature" introduced in sec. 2.5.1, and goes to infinity as the spectral width of the light decreases (since the entropy goes to zero in this case, as noted in [15]).[36] We are led to consider three temperatures related to the aperture lamp [57], $T_{\rm phosphor}$ = the temperature of the phosphor, $T_{\rm output}$ = the effective temperature of the output visible light from the lamp according to eq. (38), and $T_{\rm pump}$ = the effective temperature of the input/pump ultraviolet light. Then, energy conservation together with the 2nd law of thermodynamics imply that the efficiency $\eta$ of the lamp is bounded by

$$\eta \equiv \frac{P_0}{P_{\rm in}} \leq \frac{1 - T_{\rm phosphor}/T_{\rm input}}{1 - T_{\rm phosphor}/T_{\rm output}} \tag{39}$$

In typical lamps, $T_{\rm phosphor} \ll T_{\rm input}$ and $T_{\rm phosphor} \ll T_{\rm output}$, so the limit (39) becomes $\eta \leq 1$, which is hardly surprising. If the pump beam has a very narrow spectrum but the output beam does not, then $T_{\rm input} \to \infty$ while $T_{\rm output}$ is finite and it is possible that $\eta > 1$. This

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[33]For illustrations of this fact in experiments, see, for example, [51, 52].

[34]The surface of the glass envelope of the blacklight of sec. 2.5.1 is at room temperature, not at the "brightness temperature" of 7700K. This indicates that the exchange of energy between the light and the glass envelope is insufficient for the light to be in thermal equilibrium with its environment.

[35]An optical heat pump based on a light-emitting diode has been reported in [53]. See also [54].

[36]The effective temperature (38) of the light emitted by a blackbody is the same as the temperature of the blackbody, as shown in [55] by a limiting argument as the excess $U$ and $S$ go to zero.

corresponds to so-called anti-Stokes behavior of the phosphor in which some of its thermal energy is transfered into the output light along with energy from the input light [58, 59].[37]

## 2.7   An Optical Water Heater

Instead of the output power $P_0$ of the lamp appearing at light of an effective temperature $T_1 \gg T_{\text{ambient}}$, it would be interesting if this power could be used to, say, heat water to temperature $T_2 > T_{\text{ambient}}$. A possible configuration for this is sketched below, in which phosphor 1 exists over the entire inner surface of a cylinder and a smaller water tube coated with phosphor 2 is coaxial with the outer cylinder.
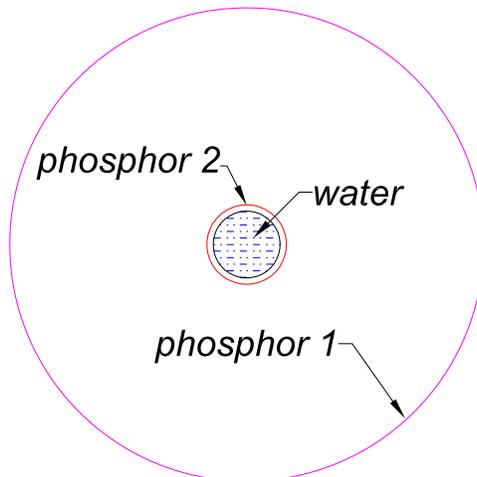


We assume that phosphor 1 is like that considered in sec. 2.5.1, and has the additional feature that it reflects light of other wavelengths that those which can excite the phosphor. That is, light incident on phosphor 1 of any wavelength does not change the thermodynamic temperature of the phosphor (or the material of the cylinder on which that phosphor is coated).

In contrast, phosphor 2 should be something close to an ideal blackbody, which absorbs all incident light while emitting light with a Planck spectrum of temperature $T_2$, which can also be the temperature of the water inside the tube on which phosphor 2 is coated.

In this idealized scenario, all output power of the lamp is used to heat the water, whose output temperature is related to its input temperature by

$$T_{\text{out}} = T_{\text{in}} + \frac{P_0}{C \, dM/dt} \, , \tag{40}$$

where $C \approx 2250$ kJ/kg is the heat capacity of water, and $dM/dt$ is the mass flow rate of water through the tube.

*However, such an optical water heater is a somewhat indirect way of converting the input electrical power that excites phosphor 1 into heated water, since heating of water by current*

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[37]An ideal blackbody can absorb light at a single wavelength and re-emit light at all wavelengths, and so exhibits both Stokes and anti-Stokes shifts (towards longer and shorter wavelengths, respectively). If the phosphor is an ideal blackbody, the output power $P_0$ (now given by the Stefan-Boltzmann law, and with effective temperature $T_{\text{output}} = T_{\text{phosphor}}$) comes from the thermal bath which maintains the temperature $T_{\text{phosphor}}$; the external power input $P_{\text{in}}$ is zero, such that the efficiency (39) is infinite.

*flowing through a resistor also obeys eq. (40). A possibly more interesting apparatus would be one in which the energy of sunlight incident on some collector could be used to heat water, but even here the advantage of a scheme utilizing phosphors is not obvious.*

### 2.7.1 Could an Optical Water Heater Transfer Thermal Energy from a Low- to a High-Temperature Phosphor?

Suppose in the optical water heater sketched above the energy input does not come from UV light as in a fluorescent lamp, but from a thermal bath at temperature $T_1$ which is in contact with phosphor 1. Could net (thermal) energy then be transferred from phosphor 1 via radiation to phosphor 2, when the latter is at temperature $T_2 > T_1$?

Such an energy transfer would be a violation of the 2nd law of thermodynamics, so we can safely say that the answer is no. However, the thermodynamics of phosphors is intricate, and it is not immediately evident why the hypothetical energy transfer cannot take place.[38]

*April 1, 2017. The preceeding paragraph is incorrect/too pessimistic. The rest of this section is superceded by my note [62].*

The premise of this section is that in both phosphors 1 and 2 their thermal energy can be transferred into radiant energy of some wavelengths, and *vice versa*. Radiation emitted at these wavelengths will be called thermal radiation.[39] As noted by Landau [50], the surface brightness of thermal radiation by the phosphor (at any wavelength for which the radiation process can exchange energy with the thermal energy of the phosphor) is less than or equal to that of the thermal radiation (at that wavelength) by a blackbody of the same temperature as the phosphor. An immediate consequence is that no significant thermal energy would be radiated by a room-temperature phosphor (and that the temperatures required for significant thermal radiation of visible light by any type of material exceed the boiling point of water).

According to the 2nd law of thermodynamics, if phosphor 2 has a higher thermodynamic temperature than phosphor 1, there will be a net transfer of some of the thermal energy of phosphor 2 away from this phosphor until $T_2 = T_1$. There are two possibilities for the fate of the thermal energy radiated by phosphor 2:

1. Some of the radiant energy emitted by phosphor 2 could be absorbed by phosphor 1 and transferred into thermal energy (which then "disappears" into the thermal bath which maintains temperature $T_1$).

2. If the radiant energy emitted by phosphor 2 has a wavelength for which there is no exchange of energy with the thermal energy of phosphor 1, the radiated energy remains in a nonthermal form (with respect to phosphor 1). *For example, if this radiant energy is reflected (specularly or not) by phosphor 1, and likewise reflected (specularly*

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[38]One past debate on the thermodynamics of fluorescence started with Vavilov's comment [60] on Pringsheim's 1929 paper [58], which led to responses by Pringsheim [59], Vavilov [61] and Landau [50].

[39]Such thermal radiation is limited/nonexistent in many phosphors, as considered in sec. 2.5.1, for which the "brightness temperature" of their (nonthermal) radiation can be far larger than the thermodynamic temperature of the phosphor/substrate. The phosphor might be transparent to such nonthermal light, or reflect it specularly, or absorb it and re-emit light of the same (or other) wavelength randomly.

*or not) by phosphor 2, it remains trapped in the space between the two phosphors. The energy (and the "brightness temperature") of this trapped radiation, which is not in thermal equilibrium with any material, grows with time. Meanwhile, the thermodynamic temperature of phosphor 2 drops as it radiates this energy. Or, if the light is reflected by phosphor 1 and absorbed by phosphor 2, it transfers no net energy away from phosphor 2.*

By assumption, phosphor 1 emits thermal radiation at some wavelengths that can be absorbed by phosphor 2 and for which wavelength this energy can be exchanged with the thermal energy of phosphor 2. The issue is whether the amount of radiant energy in this category which is absorbed by phosphor 2 is larger or smaller than the amount of thermal energy radiated by phosphor 2. The 2nd law of thermodynamics advises us that it is smaller, and that phosphor 2 will radiatively cool until its temperature is the same as phosphor 1.[40,41]

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[40]One can imagine hypothetical phosphors for which, say, the wavelength band for absorption is large while that for emission is small (as in the phosphor of sec. 2.5.1) and that the absorption and emission processes can exchange energy with the thermal energy of the phosphor (which does not occur in the phosphor of sec. 2.5.1), such that more radiative energy might be absorbed than emitted, thereby increasing the thermodynamic temperature of the phosphor. The lesson of the 2nd law of thermodynamics is that such hypothetical materials do not exist.

[41]So-called photonic bandgap materials [37] emit and absorb photons of energy only larger than their bandgap energy. Assuming that this emission and absorption couples to thermal processes in the material, we can consider phosphors 1 and 2 to be such materials, with bandgap energies $E_1$ and $E_2$. Then, all photons with energies greater than $\max\{E_1, E_2\}$ can be absorbed by both phosphors, and radiant energy transfer by such photons equilibrates the thermodynamic temperature of the two phosphors. Photons with energies between $\min\{E_1, E_2\}$ and $\max\{E_1, E_2\}$ are emitted and absorbed by one of the phosphors but not by the other, and do not participate in the transfer of thermal energy between the phosphors; the brightness temperature of these photons is (by Landau's theorem) less than that of the thermodynamic temperature of the phosphor that emits/absorbs them (which temperature is the same for both phosphors in the steady state).

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