

Biot-Savart *vs.* Einstein-Laub Force Law

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1 Problem

The Biot-Savart force density \mathbf{f} on a conduction-current density \mathbf{J}_{cond} in a magnetic field is commonly written (in SI units) as

$$\mathbf{f}_{\text{Biot-Savart}} = \mathbf{J}_{\text{cond}} \times \mathbf{B}, \quad (1)$$

which is verified by experiment when the current flow in a magnetic medium, provided the field \mathbf{B} used in eq. (1) is the “initial” field that would exist in the absence of the current.^{1,2,3} The extrapolation of eq. (1) to a single charge q with velocity \mathbf{v} ,

$$\mathbf{F}_{\text{Lorentz}} = q\mathbf{v} \times \mathbf{B}, \quad (3)$$

has been verified experimentally when the charge moves inside a magnetic medium for which \mathbf{B} is much larger than $\mu_0\mathbf{H}$ [14].

It is sometimes preferable that the force law be given in term of the total magnetic field on the current, in which case it is generally best (in the author’s opinion) to use the Maxwell stress tensor [3]. In 1908 Einstein and Laub [9, 15, 16] argued that the force density could be written in terms of the total \mathbf{H} field and the magnetization density \mathbf{M} (of quantum magnetic dipoles) in the medium that supports the current law as

$$\mathbf{f}_{\text{Einstein-Laub}} = \mathbf{J}_{\text{cond}} \times \mu_0\mathbf{H} + \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H}. \quad (4)$$

¹For reviews, see [1, 2, 3].

²When the medium that supports the current has uniform relative permeability μ , such that $\mathbf{B} = \mu\mu_0\mathbf{H} = \mu_0(\mathbf{H} + \mathbf{M})$, the magnetization density is $\mathbf{M} = (\mu - 1)\mathbf{H}$. Associated with this magnetization is the bound current density $\mathbf{J}_{\text{bound}} = \nabla \times \mathbf{M} = (\mu - 1)\nabla \times \mathbf{H} = (\mu - 1)\mathbf{J}_{\text{cond}}$. Hence, the total current density is $\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{cond}} + \mathbf{J}_{\text{bound}} = \mu\mathbf{J}_{\text{cond}}$, and the Biot-Savart force density can be written

$$\mathbf{f}_{\text{Biot-Savart}} = \mathbf{J}_{\text{total}} \times \mu\mathbf{H}_i \quad (\text{permeable current}). \quad (2)$$

³Biot and Savart [4, 5] had no concept of the magnetic field \mathbf{B} of an electric current I , and discussed only the force on a magnetic pole p as $p \oint I d\mathbf{l} \times \hat{\mathbf{r}}/r^2$, although not, of course, in vector form. The form (1) can be traced to Grassmann (1845) [7], still not in vector form. The vector relation $\mathbf{F}_{\text{on } 1} = \oint_1 I_1 d\mathbf{l}_1 \times \mathbf{B}_2$ at 1 appears without attribution as eq. (11) of Art. 603 of Maxwell’s *Treatise* [8], while Einstein may have been the first to call this the Biot-Savart law, in sec. 2 of [9].

Heaviside, p. 551 and 559 of [10], discussed the form $d\mathbf{F} = \rho\mathbf{E} + \mathbf{\Gamma} \times \mathbf{H}$, where ρ is the electric charge density and $\mathbf{\Gamma} = \nabla \times \mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$, where \mathbf{J} is the conduction current density \mathbf{J} and $\partial\mathbf{D}/\partial t$ is the “displacement current” density. However, the present view is that the “displacement current” does not experience a magnetic force. Lorentz himself seems to have advocated the form $q\mathbf{v} \times \mu_0\mathbf{H}$ in eq. (V), sec. 12 of [11]. See also eq. (23) of [12].

The earliest description in English of eq. (1) as the Biot-Savart law may be in sec. 7-6 of [13].

The awkward second term on the right is easily neglected,⁴ as by Einstein himself in [18], which has created the misimpression that the Einstein claimed the force density is just $\mathbf{J}_{\text{cond}} \times \mu_0 \mathbf{H}$, which conflicts with experiment [1]. Also, it might be supposed that the extrapolation of the Einstein-Laub force law (4) to a “point” charge q is $\mathbf{F}_{\text{E-L}} = q\mathbf{v} \times \mu_0 \mathbf{H}$, which conflicts with experiment [14]; however, the \mathbf{H} -field in eq. (4) is that inside the current, rather than in the surrounding medium, so when extrapolating to a single charge we must make the convention that “inside” the point charge (with unit relative permeability) the \mathbf{H} -field is actually \mathbf{B}/μ_0 of the surrounding medium, such that the Lorentz law (3) still applies.

Einstein and Laub (rightly) considered that valid force densities for steady currents (in the absence of electric fields) must predict that a system exerts no net force on itself.^{5,6} Using forms (1) and (4), deduce the total force per unit length on a straight, current-carrying wire with permanent magnetization \mathbf{M} perpendicular to the axis of the wire.

Also, deduce the force per unit length on a permeable, current-carrying wire in an external, transverse magnetic field using forms (1) and (4).

Can the conduction current be replaced by an effective magnetization in eq. (4)?

2 Solution

Aspects of the following were discussed by Gans in 1911 [23].

2.1 Self Force of a Permanently Magnetized Wire

In the absence of the wire the “initial” fields are zero, so form (1) immediately predicts there to be zero self force on the wire.

The wire has radius a , carries total current I and lies along the z -axis. The permanent magnetization \mathbf{M} is taken to be in the x -direction. The medium surrounding the wire has relative permeability μ .

The conduction-current density inside the wire is

$$\mathbf{J}_C(r < a) = \frac{I}{\pi a^2} \hat{\mathbf{z}}, \quad (5)$$

and the azimuthal magnetic field due to this current follows from Ampère’s law as

$$\mathbf{B}_C(r < a) = \mu_0 \mathbf{H}_C(r < a) = \frac{\mu_0 I r}{2\pi a^2} \hat{\boldsymbol{\theta}} = \frac{\mu_0 I}{2\pi a^2} (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}), \quad (6)$$

⁴The term $\mu_0(\mathbf{M} \cdot \nabla)\mathbf{H}$ has dubious physical justification, as mentioned in sec. 2.3.1 of [17].

⁵Systems in which both electric and magnetic fields are present can exhibit nonzero self electromagnetic forces [19], in which cases one must consider electromagnetic-field momentum, and even so-called “hidden” mechanical momentum to avoid “bootstrap spaceships” [20, 21].

⁶Accelerated charges can be subject to the so-called radiation-reaction force $q^2 \ddot{\mathbf{v}}/2c^2$, which is a self force (first noted by Lorentz [22], and so should be considered as part of the “Lorentz force law”). Not all accelerated charges are subject to the radiation reaction force, since interference of the fields of the various charges may cancel the total radiation, as for steady current loops. Also, a uniformly accelerated charge (which is a kind of steady motion) famously experiences no self/radiation-reaction force.

in a cylindrical coordinate system (r, θ, z) . The total force on the conduction current due to its own magnetic field is zero according to either of the force laws (1)-(4).

To compute the fields $\mathbf{B}_M/\mu_0 = \mathbf{H}_M + \mathbf{M}$ due to the permanent magnetization $\mathbf{M}(r < a) = M \hat{\mathbf{x}}$ we note that $\nabla \cdot \mathbf{B}_M/\mu_0 = 0 = \nabla \cdot \mathbf{H}_M + \nabla \cdot \mathbf{M}$, so we can say that $\nabla \cdot \mathbf{H}_M = -\nabla \cdot \mathbf{M} \equiv \rho_M$. For the present example the volume density ρ_M of effective magnetic charges is zero both inside and outside the wire,⁷ but there is an effective surface density of magnetic charges on the outer surface of the wire,⁸

$$\sigma_M(r = a^-) = \mathbf{M}(r = a^-) \cdot \hat{\mathbf{r}} = M \cos \theta, \quad (7)$$

and also surface density on the adjacent inner surface of the surrounding medium,

$$\sigma_M(r = a^+) = -\mathbf{M}(r = a^+) \cdot \hat{\mathbf{r}} = -(\mu - 1)\mathbf{H}(r = a^+) \cdot \hat{\mathbf{r}} = -(\mu - 1)H_{M,r}(r = a^+) \quad (8)$$

Then, since $\nabla \times \mathbf{H}_M = 0$, this field can be deduced from a scalar potential, $\mathbf{H}_M = -\nabla\Phi_M$ where the potential Φ_M has the form

$$\Phi_M(r < a) = A \frac{r}{a} \cos \theta, \quad (9)$$

$$\Phi_M(r > a) = A \frac{a}{r} \cos \theta. \quad (10)$$

The matching condition at the surface of the cylinder is

$$\begin{aligned} H_{M,r}(r = a^+) - H_{M,r}(r = a^-) &= \frac{2A}{a} \cos \theta = \sigma_M(r = a^-) + \sigma_M(r = a^+) \\ &= M \cos \theta - (\mu - 1) \frac{A}{a} \cos \theta, \end{aligned} \quad (11)$$

such that

$$A = \frac{aM}{\mu + 1}, \quad (12)$$

and the potential inside the cylinder is

$$\Phi_M(r < a) = \frac{M}{\mu + 1} r \cos \theta = \frac{M}{\mu + 1} x, \quad (13)$$

The interior fields are therefore,

$$\mathbf{H}_M(r < a) = -\frac{\mathbf{M}}{\mu + 1}, \quad \mathbf{B}_M(r < a) = \mu_0[\mathbf{H}_M(r < a) + \mathbf{M}] = \frac{\mu\mu_0\mathbf{M}}{\mu + 1}, \quad (14)$$

and the exterior fields are

$$\mathbf{H}_M(r > a) = \frac{\mathbf{B}_M(r > a)}{\mu\mu_0} = \frac{Ma^2}{(\mu + 1)r^2}(\hat{\mathbf{r}} \cos \theta + \hat{\boldsymbol{\theta}} \sin \theta) = \frac{Ma^2}{(\mu + 1)r^2}(\hat{\mathbf{x}} \cos 2\theta + \hat{\mathbf{y}} \sin 2\theta), \quad (15)$$

⁷Outside the wire, $\mathbf{B} = \mu\mu_0\mathbf{H} = \mu_0(\mathbf{H} + \mathbf{M})$, $\mathbf{M} = (\mu - 1)\mathbf{H}$, so $\nabla \cdot \mathbf{B} = 0$ implies that $\nabla \cdot \mathbf{H} = 0$ and $\rho_{m,\text{eff}} = -\nabla \cdot \mathbf{M} = 0$ here.

⁸See, for example, the Appendix of [24].

such that B_r and H_θ are continuous at the surface $r = a$.

The force \mathbf{F}_C per unit length on the conduction current \mathbf{J}_C (which has zero magnetization) due to the fields of the magnetization \mathbf{M} is then

$$\mathbf{F}_{\text{Einstein-Laub},C} = \int \mathbf{J}_C \times \mu_0 \mathbf{H}_M d\text{Vol} = -\frac{\mu_0 I M}{\mu + 1} \hat{\mathbf{y}}. \quad (16)$$

The force \mathbf{F}_M per unit length on the magnetization due to the fields of the conduction current is then

$$\mathbf{F}_{\text{Einstein-Laub},M} = \int (\mathbf{M} \cdot \nabla) \mu_0 \mathbf{H}_C d\text{Vol} = \int M \frac{\partial}{\partial x} \frac{\mu_0 I}{2\pi a^2} (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}) d\text{Vol} = \frac{\mu_0 I M}{2} \hat{\mathbf{y}}. \quad (17)$$

Thus, the total self-force, $\mathbf{F}_C + \mathbf{F}_M = (\mu - 1)\mu_0 I M \hat{\mathbf{y}}/2(\mu + 1)$, on the wire is nonzero for the Einstein-Laub form (4) if the relative permeability μ of the medium surrounding the wire is not unity.^{9,10} **Thus, the Einstein-Laub form fails to meet their own criterion for validity.**

We can also calculate the force \mathbf{F}_μ on the permeable medium at $r > a$ using the force density (4) for the exterior fields (15), noting that in the permeable medium the magnetization is given by

$$\mathbf{M} = (\mu - 1)\mathbf{H} = (\mu - 1) \left(\frac{I}{2\pi r} \hat{\boldsymbol{\theta}} + \frac{Ma^2}{(\mu + 1)r^2} (\hat{\mathbf{r}} \cos \theta + \hat{\boldsymbol{\theta}} \sin \theta) \right), \quad (19)$$

and that $\partial \hat{\mathbf{r}}/\partial \theta = \hat{\boldsymbol{\theta}}$, $\partial \hat{\boldsymbol{\theta}}/\partial \theta = -\hat{\mathbf{r}}$,

$$\begin{aligned} \mathbf{F}_{\text{E-L},\mu} &= \int_{r>a} (\mathbf{M} \cdot \nabla) \mu_0 \mathbf{H} d\text{Vol} \\ &= (\mu - 1)\mu_0 \int \left[\frac{I}{2\pi r^2} \frac{\partial}{\partial \theta} + \frac{Ma^2}{(\mu + 1)r^2} \left(\cos \theta \frac{\partial}{\partial r} + \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \right] \\ &\quad \left(\frac{I}{2\pi r} \hat{\boldsymbol{\theta}} + \frac{Ma^2}{(\mu + 1)r^2} (\hat{\mathbf{r}} \cos \theta + \hat{\boldsymbol{\theta}} \sin \theta) \right) d\text{Vol} \\ &= \frac{(\mu - 1)\mu_0 I M a^2}{2\pi(\mu + 1)} \int_a^\infty r dr \int_0^{2\pi} d\theta \frac{-\hat{\mathbf{r}} \sin \theta + \hat{\boldsymbol{\theta}} \cos \theta}{r^4} \\ &= \frac{(\mu - 1)\mu_0 I M a^2}{2\pi(\mu + 1)} \int_a^\infty \frac{dr}{r^3} \int_0^{2\pi} d\theta (-\hat{\mathbf{x}} \sin 2\theta + \hat{\mathbf{y}} \cos 2\theta) = 0. \end{aligned} \quad (20)$$

⁹For completeness, we note that the force on the magnetization due to its own field is zero using either form (1) or (4),

$$\mathbf{F}_{\text{self,Einstein},M} = \int (\mathbf{M} \cdot \nabla) \mu_0 \mathbf{H}_M d\text{Vol} = 0. \quad (18)$$

¹⁰See the Appendix for the case that the permeable medium has inner radius $b > a$.

2.2 Force on a Permeable Wire in an External Magnetic Field

Consider a current-carrying wire of relative permeability μ that is embedded in a medium of relative permeability μ' . The wire runs along the z -axis, has area A , carries conduction current

$$\mathbf{J}_{\text{cond}} = \frac{I}{A} \hat{\mathbf{z}}, \quad (21)$$

and is immersed in an “initial” (external) magnetic field

$$\mathbf{B}_i = B_0 \hat{\mathbf{x}}, \quad \mathbf{H}_i = H_0 \hat{\mathbf{x}} = \frac{B_i}{\mu' \mu_0} = \frac{B_0}{\mu' \mu_0} \hat{\mathbf{x}}. \quad (22)$$

The force per unit length on the wire according to eq. (1) is

$$\mathbf{F}_{\text{Biot-Savart}} = I \hat{\mathbf{z}} \times B_0 \hat{\mathbf{x}} = IB_0 \hat{\mathbf{y}}, \quad (23)$$

as confirmed experimentally [1]¹¹ (and which can be deduced by several other methods as reviewed in [3]).

Section 2.1 confirmed for a the original example of [9] that the Einstein-Laub force density (4) results in no net magnetic force of the system on itself. This suggests that if we write the total magnetic field as

$$\mathbf{B} = \mathbf{B}_i + \mathbf{B}_w, \quad \mathbf{H} = \mathbf{H}_i + \mathbf{H}_w. \quad (24)$$

where \mathbf{B}_w and \mathbf{H}_w are the fields induced in the system by the presence of the wire, then it would suffice to compute the force on the wire using the only the “initial” field \mathbf{H}_i . That is,

$$\mathbf{F}_{\text{Einstein-Laub}} = \mu_0 \int [\mathbf{J}_{\text{cond}} \times \mathbf{H}_i + (\mathbf{M} \cdot \nabla) \mathbf{H}_i] d\text{Vol}. \quad (25)$$

Since the “initial” magnetic field \mathbf{H}_i is uniform in space the gradient term in eq. (25) makes no contribution for any form of magnetization \mathbf{M} , and eq. (25) predicts the force per unit length on the wire to be

$$\mathbf{F}_{\text{Einstein-Laub}} = \mu_0 I H_0 \hat{\mathbf{y}} = \frac{IB_0}{\mu'} \hat{\mathbf{y}}, \quad (26)$$

which differs from the “correct” result (23) when the relative permeability μ' of the medium surrounding the wire is different from unity.¹²

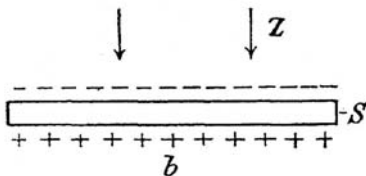
To clarify this discrepancy, we go into more detail, based on the “wire” shown in the figure below from p. 545 of [9], which is a strip of magnetic material of relative permeability μ , surrounded by a medium with relative permeability μ' .¹³ The initial/external magnetic field $\mathbf{B}_i = \mu' \mu_0 \mathbf{H}_i = B_0 \hat{\mathbf{x}}$ points downwards in the figure. The current $\mathbf{I} = I \hat{\mathbf{z}}$ is out of the

¹¹Einstein and Laub [9] were aware that eq. (23) is the experimental result. It would be interesting to know what data on this topic were available in 1908.

¹²This case was not considered in [9].

¹³In [9], $\mu' = 1$.

page, so the Biot-Savart force $IB_0 \hat{\mathbf{y}}$ per unit length due to the initial field acting on the current points to the right (the $+y$ -direction).



If the strip carried no current, and its thickness a in x is much less than its width b in y , the magnetic field inside the strip (due to the external sources) would be approximately

$$\mathbf{B}_{\text{ext,in}} = \mathbf{B}_i = B_0 \hat{\mathbf{x}}, \quad \mathbf{H}_{\text{ext,in}} = \frac{\mathbf{B}_{\text{ext,in}}}{\mu\mu_0} = \frac{B_0}{\mu\mu_0} \hat{\mathbf{x}}, \quad (27)$$

noting that the normal component of \mathbf{B} is continuous across the boundaries at $|x| = a/2$. The magnetic field inside the wire, ($|x| < a/2, |y| < b/2$), due to the current I in the wire is given approximately by

$$\mathbf{H}_{\text{wire,in}} = \frac{Ix}{ab} \hat{\mathbf{y}}, \quad \mathbf{B}_{\text{wire,in}} = \mu\mu_0 \mathbf{H}_{\text{in}} = \frac{\mu\mu_0 Ix}{ab} \hat{\mathbf{y}}. \quad (28)$$

The total magnetic field inside the wire is the sum of eqs. (27) and (28),

$$\mathbf{B}_{\text{in}} = B_0 \hat{\mathbf{x}} + \frac{\mu\mu_0 Ix}{ab} \hat{\mathbf{y}}, \quad \mathbf{H}_{\text{in}} = \frac{B_0}{\mu\mu_0} \hat{\mathbf{x}} + \frac{Ix}{ab} \hat{\mathbf{y}}. \quad (29)$$

The magnetization density \mathbf{M} inside the wire is approximately given by

$$\mathbf{M}_{\text{in}} = (\mu - 1)\mathbf{H}_{\text{in}} = (\mu - 1) \left(\frac{B_0}{\mu\mu_0} \hat{\mathbf{x}} + \frac{Ix}{ab} \hat{\mathbf{y}} \right). \quad (30)$$

Then,

$$(\mathbf{M}_{\text{in}} \cdot \nabla)\mathbf{H}_{\text{in}} = (\mu - 1) \left(\frac{B_0}{\mu\mu_0} \frac{\partial}{\partial x} - \frac{Ix}{ab} \frac{\partial}{\partial y} \right) \left(\frac{B_0}{\mu\mu_0} \hat{\mathbf{x}} + \frac{Ix}{ab} \hat{\mathbf{y}} \right) = \left(1 - \frac{1}{\mu} \right) \frac{IB_0}{ab} \hat{\mathbf{y}}. \quad (31)$$

and the total force on the wire according to the Einstein-Laub prescription is

$$\mathbf{F}_{\text{E-L}} = \mu_0 \int [\mathbf{J}_{\text{cond}} \times \mathbf{H}_{\text{in}} + (\mathbf{M}_{\text{in}} \cdot \nabla)\mathbf{H}_{\text{in}}] d\text{Vol} = \frac{IB_0}{\mu} \hat{\mathbf{y}} + \left(1 - \frac{1}{\mu} \right) IB_0 \hat{\mathbf{y}} = IB_0 \hat{\mathbf{y}}, \quad (32)$$

which agrees with the “correct” result (23).

However, the force of the wire on itself seems to be nonzero. To show this, we note that the magnetic field due to the wire is, inside the wire,

$$\mathbf{H}_w = \mathbf{H}_{\text{in}} - \mathbf{H}_i = \left(\frac{1}{\mu} - \frac{1}{\mu'} \right) \frac{B_0}{\mu_0} \hat{\mathbf{x}} + \frac{Ix}{ab} \hat{\mathbf{y}}. \quad (33)$$

Then, $(\mathbf{M}_{\text{in}} \cdot \nabla)\mathbf{H}_w = (\mathbf{M}_{\text{in}} \cdot \nabla)\mathbf{H}_{\text{in}}$, so the self force is

$$\begin{aligned} \mathbf{F}_{\text{E-L,self}} &= \mu_0 \int [\mathbf{J}_{\text{cond}} \times \mathbf{H}_w + (\mathbf{M}_{\text{in}} \cdot \nabla)\mathbf{H}_w] d\text{Vol} = \left(\frac{1}{\mu} - \frac{1}{\mu'}\right) IB_0 \hat{\mathbf{y}} + \left(1 - \frac{1}{\mu}\right) IB_0 \hat{\mathbf{y}} \\ &= \left(1 - \frac{1}{\mu'}\right) IB_0 \hat{\mathbf{y}}. \end{aligned} \quad (34)$$

We note that the sum of eqs. (26) and (34) is indeed the “correct” result (23), but the self force is nonzero when the relative permeability μ' of the medium surrounding the wire differs from unity.

Thus, both examples in the paper [9] of Einstein and Laub demonstrate that their form (4) is invalid, when applied to a situations only very slightly different from those which they considered.

In their discussion of this example, Einstein and Laub first computed “effective” magnetic charge densities inside the wire and on its surface, and then computed forces on these “effective” charges. However, their final form (4) is not obviously related to these “effective” charges (which represent yet another way of making force calculations in magnetic media, as reviewed in sec. 3.2 of [2] and sec. 7 of [3]).

2.3 Can Conduction Currents Be Replaced by an Effective Magnetization?

A broader discussion of this topic is given in [25].

In footnote 2 we remarked that magnetization \mathbf{M} is associated with an effective current density

$$\mathbf{J}_M = \nabla \times \mathbf{M}. \quad (35)$$

Can the conduction current \mathbf{J}_C that appears in eq. (4) be replaced by an effective magnetization \mathbf{M}_C such that the Einstein-Laub force density could be written

$$\mathbf{f}_{\text{Einstein-Laub}} = \mu_0(\mathbf{M}_{\text{total}} \cdot \nabla)\mathbf{H}, \quad \text{where} \quad \mathbf{M}_{\text{total}} = \mathbf{M}_{\text{quantum}} + \mathbf{M}_C, \quad (36)$$

noting that “ordinary” magnetization is actually a quantum effect not well described in detail by classical electrodynamics?

In the thought experiment of Einstein and Laub (sec. 2.1), $\mathbf{J}_C = I \hat{\mathbf{z}}/\pi a^2$, so the corresponding effective magnetization is

$$\mathbf{M}_C = \frac{Ir}{2\pi a^2} \hat{\boldsymbol{\theta}}, \quad \text{such that} \quad \mathbf{J}_C = \nabla \times \mathbf{M}_C. \quad (37)$$

The total magnetic field is

$$\mathbf{H} = \mathbf{H}_C + \mathbf{H}_M = \frac{Ir \hat{\boldsymbol{\theta}}}{2\pi a^2} - \frac{M \hat{\mathbf{x}}}{2} = \frac{I(-\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)}{2\pi a^2} - \frac{M \hat{\mathbf{x}}}{2} = \frac{Ir \hat{\boldsymbol{\theta}}}{2\pi a^2} - \frac{M(\hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta)}{2}. \quad (38)$$

The Einstein-Laub force per unit length on the conduction current is then

$$\begin{aligned}
\mathbf{F}_{\text{Einstein-Laub},C} &= \mu_0 \int (\mathbf{M}_C \cdot \nabla) \mathbf{H}_{\text{total}} d\text{Vol} \\
&= \mu_0 \int \frac{Ir}{2\pi a^2} \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{Ir \hat{\boldsymbol{\theta}}}{2\pi a^2} - \frac{M(\hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta)}{2} \right) d\text{Vol} \\
&= \frac{\mu_0 IM}{4\pi a^2} \int (\hat{\mathbf{r}} \sin \theta + \hat{\boldsymbol{\theta}} \cos \theta) d\text{Vol} = 0,
\end{aligned} \tag{39}$$

which differs from that found in eq. (16), and that on the (quantum) magnetization is (as previously found in eq. (17))

$$\begin{aligned}
\mathbf{F}_{\text{Einstein-Laub},M} &= \mu_0 \int (\mathbf{M}_M \cdot \nabla) \mathbf{H}_{\text{total}} d\text{Vol} \\
&= \mu_0 \int M \frac{\partial}{\partial x} \left(\frac{I(-\hat{\mathbf{x}} y + \hat{\mathbf{y}} x)}{2\pi a^2} - \frac{M \hat{\mathbf{x}}}{2} \right) d\text{Vol} = \frac{\mu_0 MI \hat{\mathbf{y}}}{2}.
\end{aligned} \tag{40}$$

Hence, the total self force on the wire would be nonzero in the Einstein-Laub formalism if one replaced the conduction current by an effective magnetization. We conclude that **conduction currents cannot be replaced by an effective magnetization.**

2.4 Comments

The slight extensions presented here of the original test examples of Einstein and Laub [9] of their force density (4) show that it suffers from the defect of having nonzero forces of systems on themselves.

It is not clear to the author how Einstein and Laub arrived at their expression (4). A possible derivation is given in Appendix B of [26], where the delicate issue of how to avoid nonzero self forces is not discussed. The derivation starts with the Lorentz force law written as

$$\mathbf{f} = \rho_{\text{total}} \mathbf{E}_{\text{total}} + \mathbf{J}_{\text{total}} \times \mathbf{B}_{\text{total}}. \tag{41}$$

However, it is well known¹⁴ that this version gives incorrect results if the fields \mathbf{E} and \mathbf{B} include contributions from the charge and current densities being acted upon. This defect propagates through the derivation [26] to eq. (B.7), the Einstein-Laub force density.

Some of the methods that avoid inclusion of self forces in computations of magnetic forces are reviewed in [3].

A Appendix: Variant on the Permanently Magnetized Wire

In this Appendix we consider a variant of the example in sec. 2.1 in which the wire of radius a with transverse, permanent magnetization $\mathbf{M}(r < a) = M \hat{\mathbf{x}}$ is surrounded by vacuum for $a < r < b$ and then by a medium of relative permeability μ for $r > b$.

¹⁴See, for example, sec. 4 of [2].

The conduction-current density inside the wire is, as before,

$$\mathbf{J}_C(r < a) = \frac{I}{\pi a^2}, \quad (42)$$

and the azimuthal magnetic field due to this current follows from Ampère's law as

$$\mathbf{B}_C(r < a) = \mu_0 \mathbf{H}_C(r < a) = \frac{\mu_0 I r}{2\pi a^2} \hat{\boldsymbol{\theta}} = \frac{\mu_0 I}{2\pi a^2} (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}), \quad (43)$$

in a cylindrical coordinate system (r, θ, z) .

To compute the fields $\mathbf{B}_M/\mu_0 = \mathbf{H}_M + \mathbf{M}$ due to the permanent magnetization $\mathbf{M}(r < a) = M \hat{\mathbf{x}}$ we note that $\nabla \cdot \mathbf{B}_M/\mu_0 = 0 = \nabla \cdot \mathbf{H}_M + \nabla \cdot \mathbf{M}$, so we can say that $\nabla \cdot \mathbf{H}_M = -\nabla \cdot \mathbf{M} \equiv \rho_M$. For the present example the volume density ρ_M of effective magnetic charges is zero, but there are effective surface densities of magnetic charges on the outer surface of the wire,

$$\sigma_M(r = a^-) = \mathbf{M}(r = a^-) \cdot \hat{\mathbf{r}} = M \cos \theta, \quad (44)$$

and also on the inner surface of the permeable medium at $r > b$, where $\mathbf{M} = (\mu - 1)\mathbf{H}$,

$$\sigma_M(r = b^+) = -\mathbf{M}(r = b^+) \cdot \hat{\mathbf{r}} = -(\mu - 1)\mathbf{H}(r = b^+) \cdot \hat{\mathbf{r}} = -(\mu - 1)H_{M,r}(r = b^+) \quad (45)$$

Then, since $\nabla \times \mathbf{H}_M = 0$, this field can be deduced from a (continuous) scalar potential, $\mathbf{H}_M = -\nabla \Phi_M$ where the potential Φ_M has the form

$$\Phi_M(r < a) = A \frac{r}{a} \cos \theta, \quad (46)$$

$$\Phi_M(a < r < b) = C \frac{r}{a} \cos \theta + D \frac{a}{r} \cos \theta, \quad (47)$$

$$\Phi_M(r > b) = B \frac{a}{r} \cos \theta. \quad (48)$$

Continuity of Φ_M at $r = a$ and b fixes coefficients C and D in terms of A and B , such that

$$\Phi_M(a < r < b) = \left[(B - A) ar + (b^2 A - a^2 B) \frac{a}{r} \right] \frac{\cos \theta}{b^2 - a^2}. \quad (49)$$

The matching condition at the surface $r = a$ is

$$\begin{aligned} M \cos \theta &= \sigma_M(r = a^-) = H_{M,r}(r = a^+) - H_{M,r}(r = a^-) = -\frac{\partial \Phi_M(r = a^+)}{\partial r} + \frac{\partial \Phi_M(r = a^-)}{\partial r} \\ &= \left[(3b^2 - a^2) A - 2a^2 B \right] \frac{\cos \theta}{a(b^2 - a^2)}, \end{aligned} \quad (50)$$

and that at the surface $r = b$ is

$$H_{M,r}(r = b^+) - H_{M,r}(r = b^-) = \sigma_M(r = b^+) = -(\mu - 1)H_{M,r}(r = b^+), \quad (51)$$

such that

$$\mu H_{M,r}(r = b^+) = H_{M,r}(r = b^-), \quad (52)$$

$$A = \frac{aM}{2} \left(1 - \frac{\mu - 1}{\mu + 1} \frac{a^2}{b^2} \right), \quad B = \frac{aM}{\mu + 1}, \quad (53)$$

and the potential inside the cylinder is

$$\Phi_M(r < a) = \frac{M}{2} \left(1 - \frac{\mu - 1}{\mu + 1} \frac{a^2}{b^2} \right) r \cos \theta = \frac{M}{2} \left(1 - \frac{\mu - 1}{\mu + 1} \frac{a^2}{b^2} \right) x, \quad (54)$$

The interior fields are therefore,

$$\mathbf{H}_M(r < a) = -\frac{\mathbf{M}}{2} \left(1 - \frac{\mu - 1}{\mu + 1} \frac{a^2}{b^2} \right), \quad \mathbf{B}_M(r < a) = \frac{\mu_0 \mathbf{M}}{2} \left(1 + \frac{\mu - 1}{\mu + 1} \frac{a^2}{b^2} \right), \quad (55)$$

The force \mathbf{F}_C per unit length on the conduction current \mathbf{J}_C (which has zero magnetization) due to the fields of the magnetization \mathbf{M} is then

$$\mathbf{F}_{\text{Einstein-Laub},C} = \int \mathbf{J}_C \times \mu_0 \mathbf{H}_M d\text{Vol} = -\frac{\mu_0 I M}{2} \left(1 - \frac{\mu - 1}{\mu + 1} \frac{a^2}{b^2} \right) \hat{\mathbf{y}}. \quad (56)$$

The force \mathbf{F}_M per unit length on the magnetization due to the fields of the conduction current is then

$$\mathbf{F}_{\text{Einstein-Laub},M} = \int (\mathbf{M} \cdot \nabla) \mu_0 \mathbf{H}_C d\text{Vol} = \int M \frac{\partial}{\partial x} \frac{\mu_0 I}{2\pi a^2} (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}) d\text{Vol} = \frac{\mu_0 I M}{2} \hat{\mathbf{y}}. \quad (57)$$

Thus, the total self-force, $\mathbf{F}_C + \mathbf{F}_M = (\mu - 1)\mu_0 a^2 I M \hat{\mathbf{y}} / 2(\mu + 1)b^2$, on the wire is nonzero for the Einstein-Laub form (4) if the relative permeability μ of the medium surrounding the wire is not unity. As $b \rightarrow a$ the self force approaches the value found in sec. 2.1.

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