

Rolling Off a Log with Friction

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

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1 Problem

In a variant of the famous problem of sliding on/off a frictionless cylindrical log, consider the case of a cylindrical object of mass m , radius r and moment of inertia $I = kmr^2$ that rolls down the log (of radius R) with coefficients μ_s and μ_k of static and kinetic (sliding) friction, respectively. At what angle θ to the vertical does the object lose contact with the log when starting from rest at its top, assuming that it rolls without slipping? If static friction is not large enough to insure rolling without slipping at all angles, at what angle does the object begin to slide? At what larger angle does it lose contact with the log if $\mu_k = 0$?

2 Solution

2.1 Rolling without Slipping

When the object has rolled without slipping through angle θ from the vertical all of the loss of gravitational potential energy is converted to kinetic energy,

$$mg\Delta h = mgR'(1 - \cos\theta) = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{mv^2(1+k)}{2}, \quad v^2 = \frac{2gR'(1 - \cos\theta)}{1+k}, \quad (1)$$

where g is the acceleration due to gravity, $R' = R + r$ is the radius of the circle in which the center of the object moves (when in contact with the log), v is the speed of the center of mass of the object, and angular velocity $\omega = v/r$ for rolling without slipping. The centripetal force on the object when moving in the circle of radius R' at angle θ has magnitude

$$F_c = \frac{mv^2}{R'} = mg \cos\theta - N, \quad (2)$$

where N is the normal force of the log on the object. If the object loses contact with the log at angle θ while rolling without slipping, the normal force goes to zero, and eqs. (1)-(2) tell us that

$$2(1 - \cos\theta) = \cos\theta(1+k), \quad \cos\theta = \frac{2}{3+k}. \quad (3)$$

That is, an object that rolls without slipping falls off at a large angle than one that slides without friction (and without rolling), which corresponds to the case $k = 0$ and $\cos\theta = 2/3$.

However, if the coefficient μ_s of static friction is small, the object may have started slipping before it lost contact with the log.

During rolling without slipping the tangential acceleration a_t follows from the time derivative of eq. (1),

$$mgR' \sin \theta \dot{\theta} = mg \sin \theta v = mva_t(1+k), \quad a_t = \frac{g \sin \theta}{1+k}, \quad (4)$$

noting that $\dot{\theta} = v/R'$. The tangential force $F_s \leq \mu_s N$ of static friction (that torques up the rolling object) opposes the tangential component $mg \sin \theta$ of the force of gravity on the object,

$$ma_t = \frac{mg \sin \theta}{1+k} = mg \sin \theta - F_s, \quad (5)$$

$$\frac{kmg \sin \theta}{1+k} = F_s \leq \mu_s N = \mu_s \left(mg \cos \theta - \frac{2mg(1-\cos \theta)}{1+k} \right), \quad (6)$$

recalling eqs. (1)-(2). Thus, for rolling without slipping the coefficient of static friction must satisfy

$$\mu_s \geq \frac{k \sin \theta}{(3+k) \cos \theta - 2}. \quad (7)$$

Since the denominator of eq. (6) vanishes when the object loses contact with the log according to eq. (3), the object could still be rolling without slipping at that moment only if μ_s were infinite.

Hence, the object always starts to roll with slipping before it loses contact with the log, at angle θ_s related by

$$\cos \theta_s = \frac{2 + (k/\mu_s) \sin \theta_s}{3+k}, \quad (8)$$

where

$$v_s^2 = \frac{2gR'(1-\cos \theta_s)}{1+k}, \quad (9)$$

recalling eq. (1).

2.2 Rolling with Slipping

Noting that

$$a_t = \frac{dv}{dt} = \frac{dv}{d\theta} \frac{d\theta}{dt} = \frac{v}{R'} \frac{dv}{d\theta} = \frac{1}{2R'} \frac{dv^2}{d\theta}, \quad (10)$$

and that $F_s = \mu_k N$ once the object slips, the θ -equation of motion (5) can be written

$$\frac{m}{2R'} \frac{dv^2}{d\theta} = mg \sin \theta - \mu_k N = mg(\sin \theta - \mu_k \cos \theta) + \frac{\mu_k m v^2}{R'}, \quad (11)$$

recalling the radial equation of motion (2). That is,

$$\frac{dv^2}{d\theta} = 2gR'(\sin\theta - \mu_k \cos\theta) + 2\mu_k v^2 \quad (\theta > \theta_s). \quad (12)$$

For nonzero μ_k , this equation is not integrable analytically, but in the approximation that sliding friction is negligible, we find

$$v^2 = v_s^2 + 2gR'(\cos\theta_s - \cos\theta) = 2gR' \left(\frac{1}{1+k} - \cos\theta \right). \quad (\mu_k = 0, \theta > \theta_s). \quad (13)$$

Using this in the radial equation of motion (2), we find that the normal force vanishes for

$$\cos\theta = \frac{2}{3+3k} < \frac{2}{3+k} < \frac{2}{3} \quad (\mu_s > 0, \mu_k = 0). \quad (14)$$

Hence, in the case of rolling without slipping up to angle θ_s (given by eq. (8)), followed by slipping without friction thereafter, the object loses contact with the log at a larger angle θ , related by eq. (14), than in the famous case of a point mass that slides without friction ($\cos\theta = 2/3$), and also at a larger angle than eq. (3) if the object could roll without slipping until it loses contact ($\cos\theta = 2/(3+k)$).