Unipolar Induction
via a Rotating, Conducting, Magnetized Cylinder

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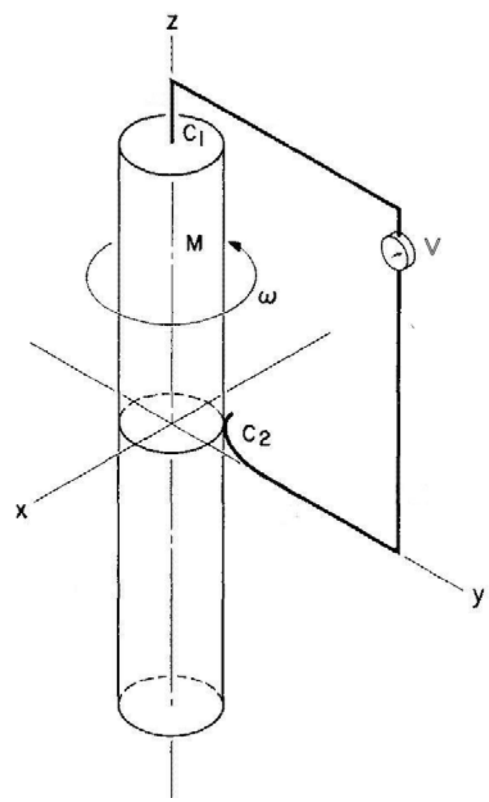
(August 17, 2008; updated November 13, 2012)

1 Problem

A conducting cylinder of radius R with permanent magnetization density $\mathbf{M}_0$ parallel to its axis when at rest is rotated about that axis with angular velocity $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ with respect to the lab frame. A voltmeter with very high internal resistance is connected to the rotating cylinder via wires with sliding contacts, one of which (C_1) is on the axis of the cylinder and the other (C_2) is on the circumference, as shown below.



Deduce the voltage V observed on the voltmeter by a lab-frame analysis as well as by an analysis in the rotating frame. You may assume that the velocity ωR is small compared to the speed of light c. Comment on the electric polarization density $\mathbf{P}$ in the cylinder should it have (relative) permittivity ϵ that differs from unity.

This configuration of unipolar induction was first considered by Faraday in 1851 [1], who also considered the case of the magnetized cylinder at rest while the voltmeter and contact wires rotated around the axis of the cylinder.

2 Solution

2.1 Analysis Using a Comoving Inertial Frame

As discussed in [2, 3, 4, 5], the best approach to an understanding of lab-frame electrodynamics of a rotating system is via a comoving inertial frame corresponding to some point in the rotating system.

We follow Minkowski [2] in arguing that the local magnetization at a point P in the rotating cylinder equals the rest value $\mathbf{M}_0$ according to an observer in the inertial frame that is instantaneously comoving with point P. That is $\mathbf{M}^\star = \mathbf{M}_0$, where the superscript * indicates quantities observed in the comoving inertial frame.

Similarly, we expect that the electric polarization $\mathbf{P}^\star$ near point P in the comoving inertial frame equals that of the magnetized cylinder in an inertial rest frame, namely $\mathbf{P}^\star = 0$.

Writing $\mathbf{v}$ as the velocity of point P in the lab frame, the field transformations to the comoving inertial frame are [6] (see also [7]), in Gaussian units and to order v/c where c is the speed of light in vacuum,

$$\mathbf{B}^\star = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}, \qquad \mathbf{D}^\star = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{E}^\star = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{H}^\star = \mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D},$$
$$\mathbf{M}^\star = \mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P}, \qquad \mathbf{P}^\star = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \tag{1}$$

and the inverse transformations are

$$\mathbf{B} = \mathbf{B}^\star + \frac{\mathbf{v}}{c} \times \mathbf{E}^\star, \qquad \mathbf{D} = \mathbf{D}^\star - \frac{\mathbf{v}}{c} \times \mathbf{H}^\star, \qquad \mathbf{E} = \mathbf{E}^\star - \frac{\mathbf{v}}{c} \times \mathbf{B}^\star, \qquad \mathbf{H} = \mathbf{H}^\star + \frac{\mathbf{v}}{c} \times \mathbf{D}^\star,$$
$$\mathbf{M} = \mathbf{M}^\star - \frac{\mathbf{v}}{c} \times \mathbf{P}^\star, \qquad \mathbf{P} = \mathbf{P}^\star + \frac{\mathbf{v}}{c} \times \mathbf{M}^\star. \tag{2}$$

We now find the lab-frame polarization and magnetization densities inside the rotating cylinder to be

$$\mathbf{P} = \frac{\mathbf{v}}{c} \times \mathbf{M}_0, \quad \mathbf{M} = \mathbf{M}_0. \tag{3}$$

As a next step we deduce the magnetic fields $\mathbf{B}$ and $\mathbf{H}$ in the lab frame from the magnetization density $\mathbf{M} = \mathbf{M}_0$ along the axis of the cylinder. Formally, the magnetic field $\mathbf{B}$ can be deduced from a vector potential $\mathbf{A}$,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{4}$$

where in the absense of free currents the vector potential is related by

$$\mathbf{A} = \frac{1}{c} \int \frac{c\boldsymbol{\nabla} \times \mathbf{M}}{\mathcal{R}}\, d\text{Vol} + \frac{1}{c} \oint_{\text{surface}} \frac{c\mathbf{M} \times d\mathbf{Area}}{\mathcal{R}}\,, \tag{5}$$

while the $\mathbf{H}$ field can be deduced (in the absense of free currents) from a magnetic scalar potential ϕ_M according to

$$\mathbf{H} = -\boldsymbol{\nabla}\phi_M, \tag{6}$$

and

$$\phi_M = \int \frac{-\boldsymbol{\nabla} \cdot \mathbf{M}}{\mathcal{R}}\, d\text{Vol} + \oint_{\text{surface}} \frac{\mathbf{M} \cdot d\mathbf{Area}}{\mathcal{R}}\,. \tag{7}$$

In the present example with uniform magnetization only the surface integrals contribute to eqs. (5) and (7), where the source is the magnetization surface current density on the cylindrical surface in eq. (5), and a uniform surface density of magnetic poles on the flat end surfaces in eq. (7). For a long cylinder the interior $\mathbf{B}$ field is essentially uniform,

$$\mathbf{B}_{\rm in} \approx 4\pi \mathbf{M}_0, \qquad \text{and} \qquad \mathbf{H}_{\rm in} = \mathbf{B} - 4\pi \mathbf{M}_0 \approx 0 \qquad \text{(long cylinder)}, \tag{8}$$

while for a thick disk the interior $\mathbf{H}$ field is essentially uniform,

$$\mathbf{H}_{\rm in} \approx -4\pi \mathbf{M}_0, \qquad \text{and} \qquad \mathbf{B}_{\rm in} = \mathbf{H} + 4\pi \mathbf{M} \approx 0 \qquad \text{(thin disk)}. \tag{9}$$

In all cases the conduction electrons must be at rest with respect to the rotating cylinder, which implies that the interior electric field vanishes in the comoving frame, $0 = \mathbf{E}^\star_{\rm in} = \mathbf{E}_{\rm in} + \mathbf{v}/c \times \mathbf{B}_{\rm in}$, recalling eq. (2). Thus,

$$\mathbf{E}_{\rm in} = -\frac{\mathbf{v}}{c} \times \mathbf{B}_{\rm in}. \tag{10}$$

A unipolar generator requires a nonzero internal electric field, so that there is a nonzero electric potential difference between the sliding contacts on the rotating disk. Hence, a thin disk does not make a good unipolar generator (although this is often used in "cartoons" of these devices).

We restrict our discussion in the rest of this note to the case (8) of a long cylinder.

Then,

$$\mathbf{E}_{\rm in} = -\frac{\mathbf{v}}{c} \times 4\pi \mathbf{M}_0 = -\frac{\boldsymbol{\omega} \times \mathbf{r}}{c} \times 4\pi \mathbf{M}_0 = -\frac{4\pi\omega M_0}{c}\mathbf{r}_\perp, \tag{11}$$

and the electric potential inside the rotating cylinder can be written,

$$V_{\rm in} = -\frac{2\pi\omega M_0}{c} r_\perp^2, \tag{12}$$

The potential difference between a point on the cylindrical surface, at radius R, and one on the axis is

$$\Delta V = \frac{2\pi R\omega M_0}{c}. \tag{13}$$

For completeness, we note that the polarization density inside the cylinder is, recalling eq. (3),

$$\mathbf{P}_{\rm in} = \frac{\mathbf{v}}{c} \times \mathbf{M}_0 = \frac{\omega M_0}{c}\mathbf{r}_\perp, \tag{14}$$

so the bound volume and surface charge densities (on the cylindrical surface) are

$$\rho_{\rm bound} = -\boldsymbol{\nabla} \cdot \mathbf{P} = -\frac{2\omega M_0}{c}, \qquad \text{and} \qquad \sigma_{\rm bound} = \mathbf{P}(R^-) \cdot \hat{\mathbf{r}}_\perp = \frac{\omega R M_0}{c} \tag{15}$$

with no surface charge on the flat ends of the cylinder. The total bound charge is, of course, zero. Finally, the electric displacement is

$$\mathbf{D}_{\rm in} = \mathbf{E}_{\rm in} + 4\pi \mathbf{P}_{\rm in} = 0. \tag{16}$$

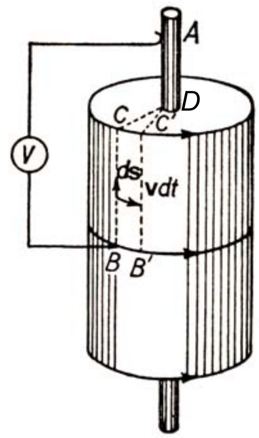
This is not a trivial result, and in the related case of a rotating, conducting magnetized sphere the electric displacement is nonzero inside the sphere [8].

2.2 Analysis in the Lab Frame

The analysis in the lab frame follows Chap. E III of [9], where it is naively assumed that the magnetization of the rotating cylinder in the lab frame is $\mathbf{M}_0$. See also [11]. Strictly speaking, the analysis of sec. 2.2 does not hold without having first made the arguments of sec. 2.1.

2.2.1 Analysis via Faraday's Law

The current in the circuit ABCDA in the figure below is negligible because of the high resistance of the voltmeter, so the resistive voltage drop in the circuit can be ignored.



Then, the reading V on the voltmeter equals the electromotive force around the circuit. According to Faraday's law, the electromotive force in the lab frame is given by

$$V = \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{1}{c}\frac{d\Phi_B}{dt}\,, \tag{17}$$

in Gaussian units, where $\Phi_B = \int_{\rm loop} \mathbf{B} \cdot d\mathbf{Area}$ is the magnetic flux linked by the loop ABCDA. Since the magnetic field $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = 4\pi M_0\,\hat{\mathbf{z}}$ inside the (long) cylinder has no azimuthal component in this example, it might seem that $\Phi_B = 0$ and hence, $V = 0$. However, because the cylinder is rotating, we can argue that the portion BCD of the loop deforms into BB′C′D during time dt where the arc BB′ has length $v\,dt = \omega R\,dt$. Hence, the flux through the loop increases by amount $\omega R^2 B\,dt/2$ during time dt, and the voltage according to eq. (1) is

$$V = -\frac{\omega R^2 B}{2c} = -\frac{2\pi\omega R^2 M_0}{c}\,. \tag{18}$$

The negative sign of the voltage (18) means that it is higher at points B and C than at points A and D. We infer that there is an inward radial electric field E_r inside the cylinder such that $V = \int_0^R E_r\,dr$, and hence

$$\mathbf{E}_{\rm in} = -\frac{\omega R B}{c}\,\hat{\mathbf{r}} = -\frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{\mathbf{v}}{c} \times 4\pi\mathbf{M}_0, \tag{19}$$

as found in eq. (10).

2.2.2 Analysis Using the Lorentz Force Law

Another well-known analysis in the lab frame notes that the conduction electrons in the rotating cylinder have no radial motion since no current flows in the circuit. These electrons, of charge e, experience a Lorentz force

$$F = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{20}$$

which must equal the centripetal force $-m\omega^2\,\mathbf{r}$ required for the charges to undergo uniform circular motion. The usual approximation is that this centripetal force is negligible on the scale of $e\mathbf{E}$, so that the electric field inside the rotating cylinder must be

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{\mathbf{v}}{c} \times 4\pi\mathbf{M}_0, \tag{21}$$

as found previously, if we again assume that $\mathbf{B} = 4\pi\mathbf{M}_0$,

2.3 Analysis in the Rotating Frame

The principles of electrodynamics in a rotating frame are summarized in the Appendix.

We cannot assume without question that the magnetization of the cylinder is $\mathbf{M}_0$ according to an observer at rest in the rotating frame. The best strategy is to use the comoving analysis of sec. 2.1 to identify the fields in the lab frame, and then use the transformations (28)-(29) to find the fields in the rotating frame, which we designate with a $'$:

$$\mathbf{B}' = 4\pi\mathbf{M}_0, \quad \mathbf{D}' = 0, \quad \mathbf{E}' = 0, \quad \mathbf{H}' = 0, \quad \mathbf{P}' = 0, \quad \mathbf{M}' = \mathbf{M}_0. \tag{22}$$

In the spirit of sec. 2.2, we might have naively assumed these results to be obvious.

We can now consider Maxwell's equations (34)-(37) for $\mathbf{D}'$ and $\mathbf{H}'$ in the rotating frame. In the present example there are no free sources for $\mathbf{D}'$ or $\mathbf{H}'$, and also no "other" sources according to eqs. (38)-(39). Thus, it is consistent with Maxwell's equations in the rotating frame that $\mathbf{D}' = 0 = \mathbf{H}'$. Then, using $\mathbf{P}' = 0$ and $\mathbf{M}' = \mathbf{M}_0$ we have that $\mathbf{E}' = 0$ and $\mathbf{B}' = 4\pi\mathbf{M}_0$.

Alternatively, we can consider Maxwell's equations (40)-(41) for $\mathbf{E}'$ and $\mathbf{B}'$. On examining the extensive list (42)-(45) of possible sources in the rotating frame, we see that the eqs. (40)-(41) reduce to

$$\boldsymbol{\nabla}' \cdot \mathbf{E}' = 0, \qquad \boldsymbol{\nabla}' \times \mathbf{B}' = \boldsymbol{\nabla}' \times 4\pi\mathbf{M}_0, \tag{23}$$

so that we again find $\mathbf{E}' = 0$ and $\mathbf{B}' = 4\pi\mathbf{M}_0$.

Transforming the fields from the rotating frame back to the lab frame we again obtain the results of eq. (3).[1]

Although the electric polarization, $\mathbf{P}' = 0$, vanishes in the rotating frame (since this could only be due to a moving magnetization in this example), the bound charge density (30) is nonzero,

$$\rho'_{\rm bound} = -\boldsymbol{\nabla}' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \mathbf{M}' = -\frac{2\omega M_0}{c} = \rho_{\rm bound}\,, \tag{24}$$

[1]In particular, we find it completely consistent to use the transformation $\mathbf{P} = \mathbf{P}' + \mathbf{v}/c \times \mathbf{M}'$ from the rotating frame to the lab frame, despite a claim to the contrary in [15].

recalling eq. (15). Similarly, there is a bound surface charge density on the outer circumference of the cylinder in the rotating frame given by

$$\sigma'_{\rm bound} = \frac{\omega R M'}{c} = \frac{\omega R M_0}{c} = \sigma_{\rm bound}\,, \tag{25}$$

also recalling eq. (15).

A Summary of the Principles of Electrodynamics in a Rotating Frame

For reference, we reproduce the principles of electrodynamics in the frame of a slowly rotating medium where ϵ and μ differ from unity.[2,3]

The (cylindrical) coordinate transformation is

$$r' = r, \qquad \phi' = \phi - \omega t, \qquad z' = z, \qquad t' = t, \tag{26}$$

where quantities in observed in the rotating frame are labeled with a $'$. The transformations of charge and current density are

$$\rho' = \rho, \qquad \mathbf{J}' = \mathbf{J} - \rho\,\mathbf{v}, \tag{27}$$

where $\mathbf{v}$ ($v \ll c$) is the velocity with respect to the lab frame of the observer in the rotating frame. The transformations of the electromagnetic fields are

$$\mathbf{B}' = \mathbf{B}, \qquad \mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{H}' = \mathbf{H}. \tag{28}$$

The transformations of the electric and magnetic polarizations are

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \qquad \mathbf{M}' = \mathbf{M}, \tag{29}$$

if we regard these polarizations as defined by $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{B}' = \mathbf{H}' + 4\pi\mathbf{M}'$.

The lab-frame bound charge and current densities $\rho_{\rm bound} = -\boldsymbol{\nabla}\cdot\mathbf{P}$ and $\mathbf{J}_{\rm bound} = \partial\mathbf{P}/\partial t + c\,\boldsymbol{\nabla}\times\mathbf{M}$ transform to

$$\rho'_{\rm bound} = -\boldsymbol{\nabla}'\cdot\mathbf{P}' - \frac{2\boldsymbol{\omega}\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\mathbf{M}', \tag{30}$$

$$\mathbf{J}'_{\rm bound} = \frac{\partial\mathbf{P}'}{\partial t'} + c\,\boldsymbol{\nabla}'\times\mathbf{M}' + \mathbf{v}(\boldsymbol{\nabla}'\cdot\mathbf{P}') + \frac{\mathbf{v}}{c}\times\frac{\partial\mathbf{M}'}{\partial t'} + \boldsymbol{\omega}\times\mathbf{P}' - \omega\frac{\partial\mathbf{P}'}{\partial\phi'}\,. \tag{31}$$

Force $\mathbf{F}$ is invariant under the transformation (26). In particular, a charge q with velocity $\mathbf{v}_q$ in the lab frame experiences a Lorentz force in the rotating frame given by

$$\mathbf{F}' = q\left(\mathbf{E}' + \frac{\mathbf{v}'_q}{c}\times\mathbf{B}'\right) = q\left(\mathbf{E} + \frac{\mathbf{v}_q}{c}\times\mathbf{B}\right) = \mathbf{F}, \tag{32}$$

[2]This Appendix is from sec. 2.2.5 of [5].

[3]This case is discussed most thoroughly by Ridgely [13, 14], but primarily for the interesting limit of steady charge and current distributions.

where $\mathbf{v}'_q = \mathbf{v}_q - \mathbf{v}$. Similarly, the Lorentz force density $\mathbf{f}'$ on charge and current densities in the rotating frame is

$$\mathbf{f}' = \rho'\mathbf{E}' + \frac{\mathbf{J}'}{c} \times \mathbf{B}' = (\rho'_{\rm free} + \rho'_{\rm bound})\,\mathbf{E}' + \frac{\mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm bound}}{c} \times \mathbf{B}'. \tag{33}$$

Maxwell's equations in the rotating frame can be written

$$\boldsymbol{\nabla}' \cdot \mathbf{B}' = 0, \tag{34}$$

$$\boldsymbol{\nabla}' \cdot \mathbf{D}' = 4\pi\rho'_{\rm free,total} = 4\pi\left(\rho'_{\rm free} + \rho'_{\rm other}\right), \tag{35}$$

$$\boldsymbol{\nabla}' \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial ct'} = 0, \tag{36}$$

$$\boldsymbol{\nabla}' \times \mathbf{H}' - \frac{\partial \mathbf{D}'}{\partial ct'} = \frac{4\pi}{c}\mathbf{J}'_{\rm free,total} = \frac{4\pi}{c}\left(\mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm other}\right), \tag{37}$$

where $\rho'_{\rm free} = \rho_{\rm free}$ and $\mathbf{J}'_{\rm free} = \mathbf{J}_{\rm free} - \rho_{\rm free}\mathbf{v}$ are the free charge and current densities, and the "other" charge and current densities that appear to an observer in the rotating frame are

$$\rho'_{\rm other} = -\frac{\mathbf{v}\cdot\mathbf{J}'_{\rm free}}{c^2} + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial \mathbf{D}'}{\partial ct'}, \tag{38}$$

$$\mathbf{J}'_{\rm other} = \rho'_{\rm free}\mathbf{v} + \boldsymbol{\omega} \times \frac{\mathbf{D}'}{4\pi} - \frac{\omega}{4\pi}\frac{\partial \mathbf{D}'}{\partial \phi'} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{H}'}{\partial t'}. \tag{39}$$

The "other" charge and current distributions are sometimes called "fictitious" [12], but we find this term ambiguous. For an example with an "other" charge density $\boldsymbol{\omega}\cdot\mathbf{H}'/2\pi c$ in the rotating frame, see [16].

Maxwell's equations can also be expressed only in terms of the fields $\mathbf{E}'$ and $\mathbf{B}'$ and charge and current densities associated with free charges as well as with electric and magnetic polarization:

$$\boldsymbol{\nabla}' \cdot \mathbf{E}' = 4\pi\rho'_{\rm total}, \tag{40}$$

and

$$\boldsymbol{\nabla}' \times \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial ct'} = \frac{4\pi}{c}\mathbf{J}'_{\rm total}, \tag{41}$$

where

$$\begin{aligned}
\rho'_{\rm total} &= \rho'_{\rm free} - \frac{\mathbf{v}}{c^2}\cdot\mathbf{J}'_{\rm free} - \boldsymbol{\nabla}'\cdot\mathbf{P}' + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial \mathbf{D}'}{\partial ct'} \\
&= \rho'_{\rm free,total} - \boldsymbol{\nabla}'\cdot\mathbf{P}' \\
&= \rho'_{\rm free} + \rho'_{\rm bound} + \rho'_{\rm more}, \qquad (42) \\
\rho'_{\rm more} &= -\frac{\mathbf{v}}{c^2}\cdot\left(\mathbf{J}'_{\rm free} + \frac{\partial \mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}'\times\mathbf{M}'\right) + \frac{\boldsymbol{\omega}\cdot\mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial \mathbf{E}'}{\partial ct'}, \qquad (43) \\
\mathbf{J}'_{\rm total} &= \mathbf{J}'_{\rm free} + \frac{\partial \mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}'\times\mathbf{M}' + \rho'_{\rm free}\mathbf{v} + \boldsymbol{\omega}\times\frac{\mathbf{D}'}{4\pi} - \frac{\omega}{4\pi}\frac{\partial \mathbf{D}'}{\partial \phi'} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial \mathbf{H}'}{\partial t'} \\
&= \mathbf{J}'_{\rm free,total} + \frac{\partial \mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}'\times\mathbf{M}'
\end{aligned}$$

$$= \mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm bound} + \mathbf{J}'_{\rm more}\,, \tag{44}$$

$$\mathbf{J}'_{\rm more} = \mathbf{v}\left(\rho'_{\rm free} - \boldsymbol{\nabla}'\cdot\mathbf{P}' - \frac{2\boldsymbol{\omega}\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\mathbf{M}'\right) + \boldsymbol{\omega}\times\frac{\mathbf{E}'}{4\pi} - \frac{\omega}{4\pi}\frac{\partial\mathbf{E}'}{\partial\phi'} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial\mathbf{B}'}{\partial t'}\,. \tag{45}$$

The contribution of the polarization densities to the source terms in Maxwell's equations in much more complex in the rotating frame than in the lab frame.Because of the "other" source terms that depend on the fields in the rotating frame, Maxwell's equations cannot be solved directly in this frame. Rather, an iterative approach is required in general.

The constitutive equations for linear isotropic media at rest in the rotating frame are

$$\mathbf{D}' = \epsilon\mathbf{E}', \qquad \mathbf{B}' = \mu\mathbf{H}' - (\epsilon\mu - 1)\frac{\mathbf{v}}{c}\times\mathbf{E}', \tag{46}$$

in the rotating frame, and

$$\mathbf{D} = \epsilon\mathbf{E} + (\epsilon\mu - 1)\frac{\mathbf{v}}{c}\times\mathbf{H}, \qquad \mathbf{B} = \mu\mathbf{H} - (\epsilon\mu - 1)\frac{\mathbf{v}}{c}\times\mathbf{E}, \tag{47}$$

in the lab frame. The lab-frame constitutive equations (47) are the same as for a nonrotating medium that moves with constant velocity $\mathbf{v}$ with respect to the lab frame.

We can also write the constitutive equations (46) for a linear isotropic medium in terms of the fields $\mathbf{B}'$, $\mathbf{E}'$, $\mathbf{P}'$ and $\mathbf{M}'$ by noting that $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}' - 4\pi\mathbf{M}'$, so that

$$\mathbf{P}' = \frac{\epsilon - 1}{4\pi}\mathbf{E}',$$

$$\mathbf{M}' = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c}\times\frac{\mathbf{E}'}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c}\times\mathbf{P}'. \tag{48}$$

Similarly, the constitutive equations (47) in the lab frame can be written to order v/c as

$$\mathbf{P} = \frac{\epsilon - 1}{4\pi}\mathbf{E} + \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c}\times\frac{\mathbf{B}}{4\pi} = \frac{\epsilon - 1}{4\pi}\mathbf{E} + \frac{\epsilon\mu - 1}{\mu - 1}\frac{\mathbf{v}}{c}\times\mathbf{M},$$

$$\mathbf{M} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c}\times\frac{\mathbf{E}}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c}\times\mathbf{P}. \tag{49}$$

Ohm's law for the conduction current $\mathbf{J}_C$ has the same form for a medium with velocity $\mathbf{u}'$ relative to the rotating frame as it does for a medium with velocity $\mathbf{u}$ relative to the lab frame,

$$\mathbf{J}'_C = \sigma\left(\mathbf{E}' + \frac{\mathbf{u}'}{c}\times\mathbf{B}'\right) = \sigma\left(\mathbf{E} + \frac{\mathbf{u}}{c}\times\mathbf{B}\right) = \mathbf{J}_C, \tag{50}$$

where σ is the electric conductivity of a medium at rest.

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