

Ph 406: Elementary Particle Physics

Problem Set 3

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1. Deduce the nonrelativistic form factors,

$$F(q^2) = \int \rho(r) e^{i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r}, \quad (1)$$

for the spherically symmetric charge densities with characteristic radius R ,

$$\rho(r) = \begin{cases} 3Q/4\pi R^3 & (r < R), \\ 0 & (r > R), \end{cases} \quad (2)$$

$$\rho(r) = \frac{Q}{4\pi R^2} \delta(r - R), \quad (3)$$

and

$$\rho(r) = \frac{Q}{2\pi\sqrt{2\pi}R^3} e^{-r^2/2R^2}, \quad (4)$$

all of which have total charge Q . Expand these form factors to order $(qR)^2$.

A neutral particle might have charge distributions ρ_+ and ρ_- with the above forms, but with different values of the characteristic radii R_+ and R_- .

The data are often fit to the form,¹

$$F_n(q^2) = \frac{Q}{[1 + (qR)^2]^n}, \quad (5)$$

with $n = 2$. What are the corresponding forms of the charge distributions $\rho_n(r)$ for $n = 1, 2$ and 3 ?

2. Arbitrary 2×2 Unitary Matrices and Pauli Spin Matrices

This problem concerns operators that act on 2-component spinors. Such operators can be expressed as 2×2 matrices. Operators that preserve the normalization of a state are called **unitary**.

Two of the simplest unitary operators on 2-component spinors are the identity matrix $\mathbf{I}_2 = \mathbf{I}$, and the spin-flip operator \mathbf{X} (called the NOT operator in quantum computation),

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (6)$$

¹For a review of nucleon form factors, see C.F. Perdrisat *et al.*, *Nucleon electromagnetic form factors*, Prog. Part. Nucl. Phys. **59**, 694 (2007),

http://physics.princeton.edu/~mcdonald/examples/EP/perdrisat_ppnp_59_694_07.pdf.

An arbitrary 2×2 unitary matrix \mathbf{U} can be written as

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (7)$$

where a, b, c and d are complex numbers such that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$. The decomposition (7) is somewhat trivial. Express the general unitary matrix \mathbf{U} as the sum of four unitary matrices, times complex coefficients, of which two are the classical unitary matrices \mathbf{I} and \mathbf{X} given above. Denote the “partner” of \mathbf{I} by \mathbf{Z} and the “partner” of \mathbf{X} by \mathbf{Y} such that

$$\mathbf{X}\mathbf{Y} = i\mathbf{Z}, \quad \mathbf{Y}\mathbf{Z} = i\mathbf{X}, \quad \mathbf{Z}\mathbf{X} = i\mathbf{Y}. \quad (8)$$

You have, of course, rediscovered the so-called Pauli spin matrices,^{2,3}

$$\sigma_x (= \sigma_1) = \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y (= \sigma_2) = \mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z (= \sigma_3) = \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9)$$

As usual, we define the Pauli “vector” $\boldsymbol{\sigma}$ as the triplet of matrices

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z). \quad (10)$$

Show that for ordinary 3-vectors \mathbf{a} and \mathbf{b} ,

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{I} + i \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}. \quad (11)$$

With this, show that a general 2×2 unitary matrix can be written as

$$\mathbf{U} = e^{i\delta} \left(\cos \frac{\theta}{2} \mathbf{I} + i \sin \frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \right) = e^{i\delta} e^{i\frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}}, \quad (12)$$

where δ and θ are real numbers and $\hat{\mathbf{u}}$ is a real unit vector.⁴ By the exponential $e^{\mathbf{O}}$ of an operator \mathbf{O} we mean the Taylor series $\sum_n \mathbf{O}^n / n!$ where $\mathbf{O}^0 = \mathbf{I}$.

What is the determinant of the matrix representation of \mathbf{U} ? The subset of 2×2 unitary matrices with unit determinant is called the **special unitary group** $\text{SU}(2)$. What is the version of eq. (12) that describes 2×2 special unitary operators?

You may wish to convince yourself of a factoid related to eq. (12), namely that if \mathbf{A} is a square matrix of any order such that $\mathbf{A}^2 = \mathbf{I}$, then $e^{i\theta\mathbf{A}} = \cos \theta \mathbf{I} + i \sin \theta \mathbf{A}$, provided that θ is a real number. It follows that \mathbf{A} can also be written in the exponential form

$$\mathbf{A} = e^{i\pi/2} e^{-i\frac{\pi}{2}\mathbf{A}} = e^{-i\pi/2} e^{i\frac{\pi}{2}\mathbf{A}}. \quad (13)$$

²W. Pauli, *Zur Quantenmechanik des magnetischen Elektrons*, Z. Phys. **43**, 601 (1927), http://physics.princeton.edu/~mcdonald/examples/QM/pauli_zp_43_601_27.pdf.

³The Pauli spin matrices (and the unit matrix \mathbf{I}) are not only unitary, they are also hermitian, meaning that they are identical to their adjoints: $\sigma_j^\dagger = \sigma_j$.

⁴Note that if make the replacements $\theta \rightarrow -\theta$ and $\hat{\mathbf{u}} \rightarrow -\hat{\mathbf{u}}$ we obtain another valid representation of \mathbf{U} , since the physical operation of a rotation by angle θ about an axis $\hat{\mathbf{u}}$ is identical to a rotation by $-\theta$ about the axis $-\hat{\mathbf{u}}$.

There are several unitary operators of interest, such as the Pauli matrices, that are their own inverse. If we call such an operator \mathbf{V} , then its exponential representation of \mathbf{V} can be written in multiple ways,

$$\mathbf{V} = e^{i\delta} e^{i\frac{\delta}{2}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}} = \mathbf{V}^{-1} = e^{-i\delta} e^{-i\frac{\delta}{2}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}}. \quad (14)$$

3. Give the explicit 4×4 matrix form of the four Dirac matrices γ_μ ,⁵ as well as that for $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, in their representation via the 2×2 Pauli matrices \mathbf{I} and $\boldsymbol{\sigma}_i$, $i = 1, 2, 3$,

$$\gamma_0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix}, \quad (15)$$

It should be then evident that $\text{tr}(\gamma_\mu) = 0 = \text{tr}(\gamma_5)$, where tr is the trace operator. Then, it immediately follows that $\text{tr}(\not{a}) = 0$, where $\not{a} \equiv a^\mu\gamma_\mu$ and a_μ is an arbitrary 4-vector.

Show that

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{I}_4, \quad (16)$$

where $\eta_{\mu\nu}$ has diagonal elements $1, -1, -1, -1$ and \mathbf{I}_4 is the 4×4 unit matrix,⁶ and hence that

$$\text{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}, \quad \text{and} \quad \text{tr}(\not{a}\not{b}) = 4a_\mu b^\mu \equiv 4ab. \quad (17)$$

Show also that

$$\text{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}), \quad (18)$$

and hence that

$$\text{tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(ab)(cd) - (ac)(bd) + (ad)(bc)]. \quad (19)$$

A factoid which you need not demonstrate is that the Dirac equivalent of eq. (11) is

$$\not{a}\not{b} = ab\mathbf{I}_4 + \frac{a^\mu b^\nu}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \quad (20)$$

If you think that maxtrix manipulation is the key to physics, then you might enjoy my course, **Physics of Quantum Computation**,

<http://physics.princeton.edu/~mcdonald/examples/ph410problems.pdf>.

⁵The matrices γ_μ were introduced by Dirac in the form used here, but with his γ_4 being our γ_0 , in sec. 3 of *The Quantum Theory of the Electron*, Proc. Roy. Soc. London A **117**, 610 (1928), http://physics.princeton.edu/~mcdonald/examples/QED/dirac_prsla_117_610_28.pdf.

⁶The matrix \mathbf{I}_4 is typically denoted by $\mathbf{1}$.