Ph 406: Elementary Particle Physics
Problem Set 4

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1. The form,

$$\mathsf{U} = e^{i\delta}\left(\cos\frac{\theta}{2}\,\mathbf{I} + i\sin\frac{\theta}{2}\,\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}\right) = e^{i\delta}e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \tag{1}$$

of a general 2×2 unitary matrix [(Set 2, eq. (12)] suggests that these matrices have something to do with rotations. Certainly, a matrix that describes the rotation of a vector is a unitary transformation.

A general 2-component (spinor) state $|\psi\rangle = \psi_+|+\rangle + \psi_-|-\rangle$, where $\left|\psi_+\right|^2 + \left|\psi_-\right|^2 = 1$, can also be written as,

$$|\psi\rangle = e^{i\delta}\left(\cos\theta|+\rangle + e^{i\phi}\sin\theta|-\rangle\right). \tag{2}$$

The overall phase δ has no meaning to a measurement of $|\psi\rangle$. So, it is tempting to interpret parameters θ and ϕ as angles describing the orientation in a spherical coordinate system (r, θ, ϕ) of a unit 3-vector that is associated with the state $|\psi\rangle$. The state $|+\rangle$ might then correspond to the unit 3-vector $\hat{\mathbf{z}}$ that points up along the z-axis, while $|-\rangle \leftrightarrow -\hat{\mathbf{z}}$.

However, this doesn't work! The suggestion is that the state $|+\rangle$ corresponds to angles $\theta = 0$, $\phi = 0$ and state $|-\rangle$ to angles $\theta = \pi$, $\phi = 0$. With this hypothesis, eq. (2) gives a satisfactory representation of a spin-up state as $|+\rangle$, but it implies that the spin-down state would be $-|+\rangle = e^{i\pi}$ times the spin-up state, which is not really distinct from the spin-up state.

We fix up things be writing,

$$|\psi\rangle = e^{i\delta}\left[\cos\frac{\theta}{2}|+\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\rangle\right], \tag{3}$$

and identifying angles θ and ϕ with the polar and azimuthal angles of a unit 3-vector in an abstract 3-space (sometimes called the Bloch sphere). That is, we associate the state $|\psi\rangle$ with the unit 3-vector whose components are $\psi_x = \sin\theta\cos\phi$, $\psi_y = \sin\theta\sin\phi$ and $\psi_z = \cos\theta$. Now, the associations,

$$\text{spin up} \leftrightarrow (\theta = 0, \phi = 0) \leftrightarrow |+\rangle, \qquad \text{spin down} \leftrightarrow (\theta = \pi, \phi = 0) \leftrightarrow |-\rangle, \tag{4}$$

given by eq. (3) are satisfactory.

We then infer from eq. (3) that the spin-up and spin-down states in the direction (θ, ϕ) are, to within an overall phase factor,

$$|+(\theta,\phi)\rangle \propto \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}, \qquad |-(\theta,\phi)\rangle \propto |+(\pi-\theta,\phi+\pi)\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}. \quad (5)$$

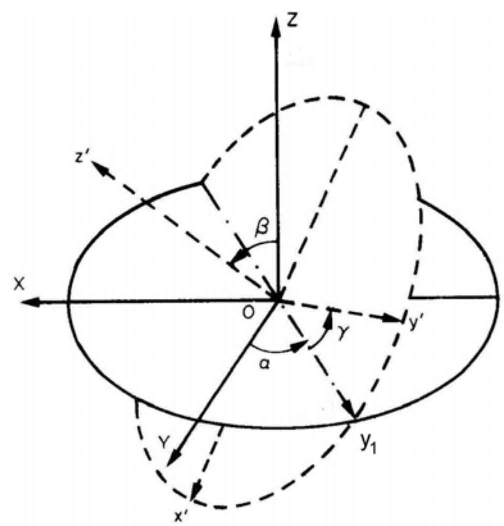
The standard form of the spin-up/down states is,

$$|+(\theta,\phi)\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \qquad |-(\theta,\phi)\rangle = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\phi/2} \\ -\cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \quad (6)$$

which is consistent with eq. (5), but perhaps does not obviously follow from it.

The Problem: Deduce the up and down 2-component spinor states along direction (θ, ϕ) in a spherical coordinate system via rotation matrices (where first a rotation is made by angle θ and then by angle ϕ).

Rotation Matrices



A general rotation in 3-space is characterized by 3 angles. We follow Euler in naming these angles as in the figure above.[1] The rotation takes the axis (x, y, z) into the axes (x', y', z') in 3 steps:

(a) A rotation by angle α about the z-axis, which brings the y-axis to the y_1 axis.

(b) A rotation by angle β about the y_1-axis, which brings the z-axis to the z'-axis.

(c) A rotation by angle γ about the z'-axis, which brings the y_1-axis to the y'-axis (and the x-axis to the x'-axis).

[1]From sec. 58 of Landau and Lifshitz, *Quantum Mechanics*.

The 2×2 unitary matrix that corresponds to this rotation (of coordinate axes) is,

$$
\begin{aligned}
\mathsf{R}(\alpha,\beta,\gamma) &= \begin{pmatrix} \cos\frac{\beta}{2}e^{i(\alpha+\gamma)/2} & \sin\frac{\beta}{2}e^{i(-\alpha+\gamma)/2} \\ -\sin\frac{\beta}{2}e^{i(\alpha-\gamma)/2} & \cos\frac{\beta}{2}e^{-i(\alpha+\gamma)/2} \end{pmatrix} \\
&= \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \\
&= \mathsf{R}_{z'}(\gamma)\mathsf{R}_{y_1}(\beta)\mathsf{R}_z(\alpha), \qquad (7)
\end{aligned}
$$

where the decomposition into the product of 3 rotation matrices[2] follows from the particular rules,

$$
\mathsf{R}_x(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}, \qquad (8)
$$

$$
\mathsf{R}_y(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}, \qquad (9)
$$

$$
\mathsf{R}_z(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \qquad (10)
$$

Convince yourself that the combined rotation (7) could also be achieved if first a rotation is made by angle γ about the z axis, then a rotation is made by angle β about the original y axis, and finally a rotation is made by angle α about the original z axis.

There is unfortunately little consistency among various authors as to the conventions used to describe rotations. I follow the notation of Barenco et al.,[3] *who appear to write eq. (7) simply as,*

$$
\mathsf{R}(\alpha,\beta,\gamma) = \mathsf{R}_z(\gamma)\mathsf{R}_y(\beta)\mathsf{R}_z(\alpha). \qquad (11)
$$

Occasionally one needs to remember that in eq. (11) the axes of the second and third rotations are the results of the previous rotation(s).

Note that according to eqs. (8)-(10),

$$
\boldsymbol{\sigma}_x = \boldsymbol{\sigma}_1 = -i\mathsf{R}_x(180^\circ), \qquad \boldsymbol{\sigma}_y = \boldsymbol{\sigma}_2 = -i\mathsf{R}_y(180^\circ), \qquad \boldsymbol{\sigma}_z = \boldsymbol{\sigma}_3 = -i\mathsf{R}_z(180^\circ), \qquad (12)
$$

and also,

$$
\boldsymbol{\sigma}_x = i\mathsf{R}_x(-180^\circ), \qquad \boldsymbol{\sigma}_y = i\mathsf{R}_y(-180^\circ), \qquad \boldsymbol{\sigma}_z = i\mathsf{R}_z(-180^\circ), \qquad (13)
$$

so that the Pauli spin matrices are equivalent to the formal matrices for 180° rotations only up to a phase factor i.

[2]The order of operations is that the rightmost rotation in eq. (7) is to be performed first.
[3]http://physics.princeton.edu/~mcdonald/examples/QM/barenco_pra_52_3457_95.pdf

Show that a more systematic relation between the Pauli spin matrices and the rotation matrices is that eqs. (8)-(10) can be written as,

$$\mathsf{R}_u(\phi) = e^{i\frac{\phi}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \tag{14}$$

which describes a rotation of the coordinate axes in Bloch space by angle ϕ about the $\hat{\mathbf{u}}$ axis (in a right-handed convention).

Rather than rotating the coordinate axes, we may wish to rotate vectors in Bloch space by an angle ϕ about a given axis $\hat{\mathbf{u}}$, while leaving the coordinate axes fixed. The operator,

$$\mathsf{R}_u(-\phi) = e^{-i\frac{\phi}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}} \tag{15}$$

performs this type of rotation. *With this in mind, you can finally solve the main problem posed on p. 2.*

2. **Helicity Conservation in High-Energy Electromagnetic Interactions** of point-like spin-1/2 particles.

   Recalling pp. 86 and 88 of Lecture 6 of the Notes, general (spin-1/2) particle 4-spinors u for plane-wave states,

$$\psi = u\, e^{-ipx} = u\, e^{-ip_\mu x^\mu}, \tag{16}$$

   with rest mass m, 3-momentum $\mathbf{p}$ and energy $E = \sqrt{p^2 + m^2}$, can be written as,

$$u = \sqrt{E+m}\begin{pmatrix} \chi \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi \end{pmatrix} = \begin{pmatrix} \sqrt{E+m}\,\chi \\ \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi \end{pmatrix} = \begin{pmatrix} \sqrt{E+m}\,\chi \\ \sqrt{E+m}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi \end{pmatrix}, \tag{17}$$

   where the 2-spinor χ obeys $\chi^\dagger\chi = 1$. Similarly, antiparticle 4-spinors v are associated with plane-wave states,[4,5]

$$\tilde{\psi} = v\, e^{ipx}, \tag{18}$$

   (note the sign change with respect to the form (16)), that can be written as,

$$v = \sqrt{E+m}\begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\tilde{\chi} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\tilde{\chi} \\ \sqrt{E+m}\,\tilde{\chi} \end{pmatrix} = \begin{pmatrix} \sqrt{E-m}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\tilde{\chi} \\ \sqrt{E+m}\,\tilde{\chi} \end{pmatrix}, \tag{19}$$

   where $\tilde{\chi}$ is a 2-spinor with $\tilde{\chi}^\dagger\tilde{\chi} = 1$.

   These states obey the Dirac equations $i\partial^\mu\gamma_\mu\psi = \not{p}\psi = m\psi$ and $i\partial^\mu\gamma_\mu\tilde{\psi} = -\not{p}\tilde{\psi} = m\tilde{\psi}$, which imply the 4-spinor equations $\not{p}u = mu$ and $-\not{p}v = mv$.

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[4] The antiparticle of particle a is often denoted as $\bar{a}$, but as $\bar{u}$ is the adjoint of a Dirac 4-spinor u, we write $\tilde{a}$ for the antiparticle of state a.

[5] Dirac interpreted his negative-energy solutions as related to "anti-electrons" on p. 52 of *Quantised Singularities in the Electromagnetic Field*, Proc. Roy. Soc. London A **133**, 60 (1931),
http://physics.princeton.edu/~mcdonald/examples/QED/dirac_prsla_133_60_31.pdf.
That paper is also noteworthy for relating the possible existence of a magnetic monopole of pole strength p to the electric charge e by $ep = \hbar/2$.

The positive and negative helicity spinor states for a particle with 3-momentum $\mathbf{p}$ in direction (θ, ϕ) are $\chi_+ = |+(\theta, \phi)\rangle$ and $\chi_- = |-(\theta, \phi)\rangle$, respectively, recalling eq. (6), while the helicity states of an antiparticle are $\tilde{\chi}_+ = |-(\theta, \phi)\rangle = \chi_-$ and $\tilde{\chi}_- = -|+(\theta, \phi)\rangle = -\chi_+$. In all cases, positive helicity means spin in the direction of momentum $\mathbf{p}$.

In the high-energy limit, these 4-spinors simplify to,

$$u \to \sqrt{E} \begin{pmatrix} \chi \\ \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}\, \chi \end{pmatrix}, \qquad v \to \sqrt{E} \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}\, \tilde{\chi} \\ \tilde{\chi} \end{pmatrix}, \tag{20}$$

Give explicit forms of the helicity spinors $u_+(\theta, \phi)$, $u_-(\theta, \phi)$, $v_+(\theta, \phi)$ and $v_-(\theta, \phi)$for (anti)particles moving and at angles (θ, ϕ) to the $+z$-axis, and also their simplification to $u_+(0)$, $u_-(0)$, $v_+(0)$ and $v_-(0)$ for motion along the z-axis in the high-energy limit.

If these are pointlike particles of charge e, their electromagnetic interaction is described by the 4-current $j_\mu = e\,\gamma_\mu$. Verify that the matrix elements $\langle \bar{u}_-(\theta)|\gamma_\mu|u_+(0)\rangle$ vanish for $\mu = 0, 1, 2, 3$, and similarly that $\langle \bar{v}_+(\theta)|\gamma_\mu|u_+(0)\rangle = 0$. Remember that $\bar{v} = v^\dagger \gamma_0$, *etc.*

Digression: Electric Charge Conjugation. The above claim that the antiparticle helicity 2-spinors $\tilde{\chi}_\pm$ are related to the particle helicity 2-spinors $\chi_\pm$ by $\tilde{\chi}_\pm = \pm\chi_\mp$ can be justified by considerations of a transformation, called electric charge conjugation with symbol C, between particles and their antiparticles (with respect to their electromagnetic interactions), such that $\tilde{\psi} = C\psi^\star$ is the antiparticle state of a spin-1/2 particle ψ.[6]

[6]That $\tilde{\psi} = C\psi^\star$ and not $\tilde{\psi} = C\psi$ follows from the sign change in the spacetime waveform between eqs. (16) and (18).

Charge conjugation leaves mass unchanged, such that a particle and its antiparticle have the same rest mass m. This was not initially understood by Dirac, who first speculated that the antiparticle of an electron is a proton, *A Theory of Electrons and Protons*, Proc. Roy. Soc. London A **126**, 360 (1930),
http://physics.princeton.edu/~mcdonald/examples/QED/dirac_prsla_126_360_30.pdf.
The charge-conjugation operator C was discussed (in a different representation, and not given a name) on p. 130 of W. Pauli, *Contributions mathématique à la théorie des matrices de Dirac*, Ann. Inst. H. Poincaré **6**, 109 (1936), http://physics.princeton.edu/~mcdonald/examples/QED/pauli_aihp_6_109_36.pdf.
The term "charge conjugation" (but with the symbol L) may have been first used in H.A. Kramers, *The use of charge conjugated wavefunctions in the hole theory of the electron*, Proc. Roy. Neder. Acad. Sci. **40**, 814 (1937), http://physics.princeton.edu/~mcdonald/examples/neutrinos/kramers_pknaw_40_814_37.pdf.
The term antimatter was introduced by Schuster in 1898, but in his vision antimatter had negative mass; *Potential Matter—A Holiday Dream*, Nature **58**, 367, 618 (1898),
http://physics.princeton.edu/~mcdonald/examples/GR/schuster_nature_58_367_98.pdf
http://physics.princeton.edu/~mcdonald/examples/GR/schuster_nature_58_618_98.pdf.
The present vision of antiparticles via electric charge conjugation of particles is perhaps closer to Kelvin's image method for a planar conductor, p. 288 of W. Thomson, *Effects of Electrical Influence on Internal Spherical and on Plane Conducting Surfaces*, Camb. Dublin Math. J. **4**, 276 (1849),
http://physics.princeton.edu/~mcdonald/examples/EM/thomson_cdmj_4_276_49.pdf.

One way to do this starts with the Dirac equation for a spin-1/2 particle state ψ,[7]

$$i\partial^\mu \gamma_\mu \psi = m\psi. \tag{21}$$

We expect that the antiparticle state $\tilde{\psi}$ also satisfies the Dirac equation,

$$i\partial^\mu \gamma_\mu \tilde{\psi} = m\tilde{\psi}. \tag{22}$$

A clever step is to take the complex conjugate of eq. (21),

$$-i\partial^\mu \gamma_\mu^* \psi^* = m\psi^*. \tag{23}$$

Applying the desired charge-conjugation operator C to this, we have,

$$-i\partial^\mu C\gamma_\mu^* \psi^* = mC\psi^* = m\tilde{\psi}. \tag{24}$$

For this to be the Dirac equation (22),[8] we require that,

$$-C\gamma_\mu^* = \gamma_\mu C. \tag{25}$$

You can verify that this implies the electric-charge-conjugation matrix operator to be,[9]

$$C = i\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}. \tag{26}$$

Then, applying the electric-charge-conjugation transformation to the particle 4-spinor u of eq. (17), we obtain (on suppression of the overall factor $\sqrt{E+m}$) the antiparticle spinor,

$$\tilde{u} = i\gamma_2 \begin{pmatrix} \chi^\star \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\chi^\star \end{pmatrix} = \begin{pmatrix} i\sigma_2 \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\chi^\star \\ -i\sigma_2\chi^\star \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}(-i\sigma_2\chi^\star) \\ -i\sigma_2\chi^\star \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\tilde{\chi} \\ \tilde{\chi} \end{pmatrix} = v, \tag{27}$$

using that fact (verify it!) that $\sigma_2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma_2$. Hence, the antiparticle 2-spinor $\tilde{\chi}$ is related to its corresponding particle 2-spinor χ by,

$$\tilde{\chi} = -i\sigma_2\chi^\star, \qquad \chi = i\sigma_2\tilde{\chi}^*. \tag{28}$$

[7]This argument follows sec. 5.4, p. 107 of F. Halzen and A.D. Martin, *Quarks and Leptons* (Wiley, 1984), http://physics.princeton.edu/~mcdonald/examples/EP/halzen_martin_84.pdf.

[8]For $\tilde{\psi} = v\,e^{ipx}$, eqs. (24)-(25) lead to the spinor form of the Dirac equation for antiparticles, $-\not{p}v = mv$.

[9]Warning: Many people write $C\gamma_0$ for the matrix C of eq. (26).

In particular, the helicity 2-spinors of eq. (6) transform under electric-charge conjugation as,

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} \quad\to\quad \tilde\chi_+ = -i\sigma_2\chi_+^* = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} = \chi_-, \tag{29}$$

$$\chi_- = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} \quad\to\quad \tilde\chi_- = -i\sigma_2\chi_-^* = \begin{pmatrix} -\cos\frac{\theta}{2}\, e^{-i\phi/2} \\ -\sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} = -\chi_+, \tag{30}$$

as claimed above.

3. The cross section for inelastic scattering of electrons off some target can be expressed in terms of two generalized structure functions $W_{1,2}(q^2,\nu)$ where $q = p_{ei} - p_{ef}$ and $\nu = q_0 = E_i - E_f$, as on p. 131, Lecture 8 of the Notes. If the inelastic scattering is due to the interaction of the virtual photon emitted by the incident electron with a spin-1/2, charge Q, mass m constituent of the target, such that the rest of the target is a "spectator" to this interaction, then the cross section is that given on p. 99, Lecture 6 of the Notes, and we infer that,[10]

$$W_1(q*2,\nu) = \frac{-q^2}{4m^2} Q^2\,\delta\left(\nu + \frac{q^2}{2m}\right), \qquad W_2(q^2,\nu) = Q^2\,\delta\left(\nu + \frac{q^2}{2m}\right). \tag{31}$$

An argument of Bjorken[11] is that the lab-frame energy difference between the initial and final electron can be written as,

$$E_i - E_f = \nu = q_0 = \frac{qP}{M}\,, \tag{32}$$

where P is the energy-momentum 4-vector of the target (of rest mass M), which is just $P = (M,0,0,0)$ in the lab frame. Then, in a frame in which the target has very high momentum, the 4-vector p of a constituent which carries (scalar) fraction x of the target's 3-momentum can be written approximately as $p \approx xP$. A consequence of this approximation is that the constituent mass m is related by $m^2 = p^2 \approx x^2P^2 = x^2M^2$, *i.e.*, that $m \approx xM$ (as appropriate for consideration of very high-energy scattering). This permits us to rewrite eq. (31) as[12]

$$W_1 = \frac{-q^2}{4M^2x^2} Q^2\,\delta\left(\nu + \frac{q^2}{2Mx}\right), \qquad W_2 = Q^2\,\delta\left(\nu + \frac{q^2}{2Mx}\right). \tag{33}$$

Supposing the constituents are distributed with the target (as viewed from a frame in which the target has high speed) with probability $f(x)\,dx$, give expressions for the generalized structure functions W_1 and W_2 in terms of a single variable x.

[10] C.G. Callan, Jr and D.J. Gross, *High-Energy Electroproduction and the Constitution of the Electric Current*, Phys. Rev. Lett. **22**, 156 (1969),
http://physics.princeton.edu/~mcdonald/examples/EP/callan_prl_22_156_69.pdf.

[11] J.D. Bjorken and E.A. Paschos, *Inelastic Electron-Proton and γ-Proton Scattering and the Structure of the Nucleon*, Phys. Rev. **185**, 1975 (1969),
http://physics.princeton.edu/~mcdonald/examples/EP/bjorken_pr_185_1975_69.pdf.

[12] A different version of this argument is given on p. 139, Lecture 8 of the Notes, where a Breit frame is used.