

Ph 406: Elementary Particle Physics

Problem Set 8

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1. The spin-1 vector mesons can be taken to have quark content: $\rho^0 = (u\bar{u} - d\bar{d})/\sqrt{2}$, $\omega^0 = (u\bar{u} + d\bar{d})/\sqrt{2}$, $\phi = s\bar{s}$, $J/\psi = c\bar{c}$, $\Upsilon = b\bar{b}$ ($V_{\text{top}} = t\bar{t}$ will not exist).

The decays $V \rightarrow e^+e^-$ proceed via a single intermediate photon, where V is a vector meson. In the quark model, this corresponds to the reaction $q\bar{q} \rightarrow \gamma \rightarrow e^+e^-$, whose cross section was discussed on p. 108, Lecture 7 of the Notes. Deduce the decay rate Γ for this by recalling (p. 13, Lecture 1 of the Notes) that

$$\text{Rate} = \Gamma_{a+b \rightarrow c+d} = N v_{\text{rel}} \sigma_{a+b \rightarrow c+d}, \quad (1)$$

where N is the number of candidate scatters per second per unit volume, and v_{rel} is the relative velocity of the initial-state particles a and b . In case of a two-particle bound state, $N = |\psi(0)|^2$ is the probability that both particles are at the origin.

Predict the decay rates to e^+e^- for the five vector mesons in the model that the strong interaction between (colored) quarks at short distances can be described by the Coulomb-like potential $V(r) \approx -4\alpha_S/3r$.

Compare with data summarized at http://pdg.lbl.gov/2013/tables/contents_tables_mesons.html.

What do we learn from this about possible energy dependence of α_S ?

2. In the vector-meson decays $V \rightarrow \pi^0\gamma$, $\eta\gamma$, the meson spin changes from 1 to 0. Hence, this must be an M1 (magnetic dipole) transition. In the quark model the M1 electromagnetic transition flips a single quark spin, but does not change quark flavor, with matrix element proportional to the relevant quark magnetic moment(s). Suppose the quarks have Dirac moments $Q_q/2m_q$ where $m_u \approx m_d \approx \frac{2}{3}m_s$. Predict the relative decay rates (not just matrix elements) to $\pi^0\gamma$ and $\eta\gamma$ for the ρ^0 , ω^0 , ϕ and J/ψ vector mesons.

Recall that in the quark model the spin-0-octet neutral mesons have quark wavefunction $\pi^0 = (u\bar{u} - d\bar{d})/\sqrt{2}$ and $\eta(548) = (2s\bar{s} - u\bar{u} - d\bar{d})/\sqrt{6}$. Compare with data summarized at http://pdg.lbl.gov/2013/tables/contents_tables_mesons.html.

3. The $\psi'(3685)$ vector meson can decay to $\chi(3415) + \gamma$. The χ particle is believed to be a 3P_0 $c\bar{c}$ state. If so, predict the angular distribution of the γ relative to the direction of the electron supposing the ψ' is produced in a colliding-beam experiment $e^+e^- \rightarrow \psi' \rightarrow \chi\gamma$. Recall that at high energies the one-photon annihilation of e^+e^- proceeds entirely via transversely polarized photons ($S_z = \pm 1$).

Crossing Symmetry

We have previously noted that the inverse processes $a + b \leftrightarrow c + d$ have common matrix elements, and that these process may proceed via single-particle exchange in any of the s -, t - or u -channels with related matrix elements. Such relations among matrix elements for related processes are sometimes called **crossing symmetry**.

In the next 3 problems you will use crossing symmetry to convert the matrix element for muon decay, $\mu \rightarrow e\bar{\nu}_e\nu_\mu$ to results for 3 related processes.

The square of the matrix element for the 4-particle vertex $\mu\nu_\mu e\nu_e$ of unpolarized particles in the Fermi theory of the weak interaction is, in terms of the particle 4-vectors p ,

$$|M|^2 = 32G_F^2(p_\mu \cdot p_{\nu_e})(p_e \cdot p_{\nu_\mu}), \quad (2)$$

where G_F is Fermi's constant, and the average over initial spins and sum over final spins is the same for all variants of the vertex.

4. Deduce the cross section for the neutrino-scattering reaction $\nu_\mu + e^- \rightarrow \mu^- + \nu_e$.

Recall that the differential cross section for 2-particle scattering $a + b \rightarrow c + d$ can be written in the center of mass frame as (p. 80, Lecture 5 of the Notes)

$$\frac{d\sigma}{d\Omega^*} = \frac{|M|^2 P_f}{64\pi^2 s P_i}, \quad (3)$$

where P_f is the momentum of the final-state particles c and d , P_i is the momentum of the initial-state particles a and b , and $s = (p_a + p_b)^2 = (p_c + p_d)^2$ is the square of the total energy in the center of mass frame. Express the cross section in terms of s , and then evaluate this in the lab frame where the electron is at rest and the muon neutrino has energy E .

5. The process $\mu^+e^- \rightarrow \mu^-e^+$ was considered by Pontecorvo in 1957 as a possible example of quantum oscillations of a two-particle system,¹ as this reaction could proceed via a two-neutrino intermediate state.

While the reaction $\mu^+e^- \rightarrow \bar{\nu}_\mu\nu_e$ is unlikely ever to be observed, it is now understood that the related reaction $e^+e^- \rightarrow \bar{\nu}_e\nu_e$ is the main source of neutrino production in supernovae, and a key process in their history.

Use a suitable variant of the matrix element (2) to deduce the cross section for $e^+e^- \rightarrow \bar{\nu}_e\nu_e$. *Work in the center-of-mass frame, and express the result in terms of the invariant s .*

¹B. Pontecorvo, *Mesonium and Antimesonium*, Sov. Phys. JETP **6**, 429 (1957),

http://physics.princeton.edu/~mcdonald/examples/EP/pontecorvo_spjjetp_6_429_57.pdf.

This landmark paper introduced the term mesonium, raised the possibility that ν_e and ν_μ are different particles, made the first speculations about neutrino oscillations, and led to the notion of conservation of lepton number, as developed by G. Feinberg and S. Weinberg, *Law of Conservation of Muons*, Phys. Rev. Lett. **6**, 381 (1961), http://physics.princeton.edu/~mcdonald/examples/EP/feinberg_prl_6_381_61.pdf.

The divergence of this cross section at low energy is avoided by Nature in that the electron and positron would not scatter but rather would bind into a positronium atom.

6. Some positronium atoms (which of the ortho- and para- states?) can decay to two neutrinos. Deduce the decay rate Γ for this by recalling (p. 13, Lecture 1 of the Notes) that

$$\text{Rate} = \Gamma_{a+b \rightarrow c+d} = N v_{\text{rel}} \sigma_{a+b \rightarrow c+d}, \quad (4)$$

where N is the number of candidate scatters per second per unit volume, and v_{rel} is the relative velocity of the initial-state particles a and b . In case of a two-particle bound state, $N = |\psi(0)|^2$ is the probability that both particles are at the origin.

Solutions

1. Vector mesons decay to e^+e^- via $q\bar{q}$ annihilation at the origin (in the rest frame of the meson) to a single photon, which materializes as an e^+e^- pair. The decay rate is then dependent on the square of the relevant quark charge(s), as well as on the square of the wavefunction at the origin, $|\psi_{q\bar{q}}(0)|^2$.

In more detail, the decay rate $\Gamma_{V \rightarrow e^+e^-}$ can be written in terms of the scattering cross section $\sigma_{q\bar{q} \rightarrow e^+e^-}$ as

$$\Gamma_{V \rightarrow e^+e^-} = N v_{\text{rel}} \sigma_{q\bar{q} \rightarrow e^+e^-}, \quad (5)$$

where N is the number of “scatters” (in which the quark and antiquark meet at the center of the vector meson “atom”) per unit volume, which can be taken as equal to the probability $|\psi_{q\bar{q}}(0)|^2$ of the atomic wave function at the origin, which for a $1/r$ potential $(4\alpha_S/3)/r$ is

$$\psi(S \text{ wave}, r) = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}, \quad \text{where} \quad a_0 = \frac{1}{(4\alpha_S/3) m_{\text{reduced}}} = \frac{3}{2\alpha_S m_q}, \quad (6)$$

recalling the Bohr atom, and the relative velocity v_{rel} of the quark and antiquark can be written as

$$v_{\text{rel}} = 2v_q = \frac{2P_i^*}{E_i^*} = 2\sqrt{1 - \frac{4m_q^2}{s}}, \quad (7)$$

where $E_i^* = \sqrt{s}/2 = m_V/2$. The velocity of the quarks inside the vector meson “atom” is low, so we need to recall the full form of the cross section, given on the bottom of p. 108, Lecture 7 of the Notes,

$$\sigma_{q\bar{q} \rightarrow e^+e^-} = \frac{4\pi\alpha^2 Q_q^2}{3s} \sqrt{\frac{1 - 4m_e^2/s}{1 - 4m_q^2/s}} (1 + 2m_e^2/s)(1 + 2m_q^2/s). \quad (8)$$

We can neglect the terms m_e^2/s for the vector mesons, but not the terms m_q^2/s . From eq. (7) we have that $2m_q^2/s = (1 - v_q^2)/2$, so the cross section can be written as

$$\sigma_{q\bar{q} \rightarrow e^+e^-} \approx \frac{4\pi\alpha^2 Q_q^2}{3sv_q} \frac{3 - v_q^2}{2} \approx \frac{2\pi\alpha^2 Q_q^2}{m_V^2 v_q}, \quad (9)$$

as $v_q \ll 1$. Before using eq. (9) in (5), we recall that the cross section for unpolarized quarks contains an initial-state spin factor $1/(2s_q + 1)^1 = 1/4$, while the decay rate Γ_V contains a spin factor $1/(2s_V + 1) = 1/3$, so we need to include a factor $4/3$ in eq. (5),

$$\Gamma_{V \rightarrow e^+e^-} \approx \frac{4}{3} 2v_q \frac{2\pi\alpha^2 Q_q^2}{m_V^2 v_q} |\psi_{q\bar{q}}(0)|^2 = \frac{16\pi\alpha^2 Q_q^2}{3m_V^2} \frac{8\alpha_S^3 m_q^3}{27\pi} \approx \frac{16\alpha^2 Q_q^2 \alpha_S^2 m_V}{81}, \quad (10)$$

where we take $m_q \approx m_V/2$ in the last step.

In the case of three colors for each quark, the probability $|\psi_{q\bar{q}}(0)|^2$ for each colored quark is 1/3 that for the case of no color, so on summing over color the result (10) is unchanged.

For the ρ^0 and ω , the effective quark charge is the quark charge weighted by the amplitudes for the $u\bar{u}$ and $d\bar{d}$ in the wavefunction. Thus, we predict,²

$$\rho^0 = \frac{u\bar{u} - d\bar{d}}{\sqrt{2}} \Rightarrow \Gamma_{\rho^0 \rightarrow e^+e^-} \propto \left| \frac{q_u - q_d}{\sqrt{2}} \right|^2 \alpha_S^3 m_\rho = \frac{1}{2} \alpha_S^3 m_\rho, \quad (11)$$

$$\omega = \frac{u\bar{u} + d\bar{d}}{\sqrt{2}} \Rightarrow \Gamma_{\omega \rightarrow e^+e^-} \propto \left| \frac{q_u + q_d}{\sqrt{2}} \right|^2 \alpha_S^3 m_\omega = \frac{1/9}{2} \alpha_S^3 m_\omega = \frac{1}{18} \alpha_S^3 m_\omega, \quad (12)$$

$$\phi = s\bar{s} \Rightarrow \Gamma_{\phi \rightarrow e^+e^-} \propto q_s^2 \alpha_S^3 m_\phi = \frac{1}{9} \alpha_S^3 m_\phi, \quad (13)$$

$$J/\psi = c\bar{c} \Rightarrow \Gamma_{J/\psi \rightarrow e^+e^-} \propto q_c^2 \alpha_S^3 m_{J/\psi} = \frac{4}{9} \alpha_S^3 m_{J/\psi}, \quad (14)$$

$$\Upsilon = b\bar{b} \Rightarrow \Gamma_{\Upsilon \rightarrow e^+e^-} \propto q_b^2 \alpha_S^3 m_\Upsilon = \frac{1}{9} \alpha_S^3 m_\Upsilon. \quad (15)$$

From http://pdg.lbl.gov/2013/tables/contents_tables_mesons.html we learn that

$$\Gamma_{\rho^0 \rightarrow e^+e^-} = 7.04 \pm 0.06 \text{ keV}, \quad (16)$$

$$\Gamma_{\omega \rightarrow e^+e^-} = 0.60 \pm 0.02 \text{ keV}, \quad (17)$$

$$\Gamma_\phi = 4.26 \pm 0.04 \text{ MeV}, \quad \frac{\Gamma_{\phi \rightarrow e^+e^-}}{\Gamma_\phi} = 2.95 \pm 0.03 \times 10^{-4},$$

$$\Gamma_{\phi \rightarrow e^+e^-} = 1.27 \pm 0.02 \text{ keV}, \quad (18)$$

$$\Gamma_{J/\psi \rightarrow e^+e^-} = 5.55 \pm 0.14 \text{ keV}, \quad (19)$$

$$\Gamma_{\Upsilon \rightarrow e^+e^-} = 1.34 \pm 0.02 \text{ keV}. \quad (20)$$

The predictions for the relative decay widths are remarkably consistent with $\alpha_S^3 m_V = \text{const.}$, *i.e.*, that the strong interaction becomes weaker at higher energy scales.

Certain considerations of broken SU(3) have led³ to the additional prediction that

$$\frac{m_\rho}{3} \Gamma_{\rho^0 \rightarrow e^+e^-} = m_\omega \Gamma_{\omega \rightarrow e^+e^-} + m_\phi \Gamma_{\phi \rightarrow e^+e^-}, \quad (21)$$

²Predictions of this type were first made by Y. Nambu and J.J. Sakurai, *Rare Decay Modes of the ω (η) Meson*, Phys. Rev. Lett. **8**, 79 (1962), http://physics.princeton.edu/~mcdonald/examples/EP/nambu_pr1_8_79_62.pdf.

The first quark-model version was by R. Van Royen and V. Weisskopf, *Hadron Decay Processes and the Quark Model*, Nuovo Cim. **50**, 617 (1967),

http://physics.princeton.edu/~mcdonald/examples/EP/vanroyen_nc_50_617_67.pdf.

The color-force wavefunction was added by T. Appelquist and H.D. Politzer, *Heavy Quarks and e^+e^- Annihilation*, Phys. Rev. Lett. **34**, 43 (1975),

http://physics.princeton.edu/~mcdonald/examples/EP/appelquist_pr1_34_43_75.pdf.

³T. Das, V.L. Mathur and S. Okubo, *Asymptotic SU(3) and Vector Meson Decays*, Phys. Rev. Lett. **19**, 470 (1967), http://physics.princeton.edu/~mcdonald/examples/EP/das_pr1_19_470_67.pdf,

J.J. Sakurai, *Vector Currents and Spectral-Function Sum Rules*, Phys. Rev. Lett. **19**, 803 (1967),

http://physics.princeton.edu/~mcdonald/examples/EP/sakurai_pr1_19_803_67.pdf.

which is better satisfied by the data than the combination (for constant $\alpha_S^3 m_V$) of eqs.(11)-(13),

$$\frac{\Gamma_{\rho^0 \rightarrow e^+ e^-}}{3} = \Gamma_{\omega \rightarrow e^+ e^-} + \Gamma_{\phi \rightarrow e^+ e^-}. \quad (22)$$

2. $V \rightarrow \pi^0 \gamma, \eta \gamma$

Since the M1 transition flips quark spin but does not change quark flavor, we immediately predict that

$$\Gamma_{\phi \rightarrow \pi^0 \gamma} = \Gamma_{J/\psi \rightarrow \pi^0 \gamma} = \Gamma_{J/\psi \rightarrow \eta \gamma} = 0. \quad (23)$$

The matrix elements for the decays $\rho^0, \omega \rightarrow \pi^0 \gamma$ have the form

$$M_{\rho^0 \rightarrow \pi^0 \gamma} = \langle (u\bar{u} - d\bar{d})/\sqrt{2} | \mu | (u\bar{u} - d\bar{d})/\sqrt{2} \rangle = \frac{\langle u\bar{u} | \mu | u\bar{u} \rangle + \langle d\bar{d} | \mu | d\bar{d} \rangle}{2}. \quad (24)$$

To evaluate the matrix element $\langle u\bar{u} | \mu | u\bar{u} \rangle$ we must note the spin structure of the quark states. In the final state $\pi^0 \gamma$ the spin component along the direction of the photon is only $S_z = \pm 1$, so the vector meson only decays to this final state if it has $S_z = \pm 1$. It suffices (since parity is conserved in this electromagnetic decay) to consider the quark state of the vector meson to be $|u\uparrow\bar{u}\uparrow\rangle$. The π^0 is spinless, so its $u\bar{u}$ quark state $(|u\uparrow\bar{u}\downarrow\rangle - |u\downarrow\bar{u}\uparrow\rangle)/\sqrt{2}$. Thus, the spin-flip matrix element is

$$\langle u\bar{u} | \mu | u\bar{u} \rangle = \langle (|u\uparrow\bar{u}\downarrow\rangle - |u\downarrow\bar{u}\uparrow\rangle)/\sqrt{2} | \mu | u\uparrow\bar{u}\uparrow \rangle \propto \frac{\mu_{\bar{u}} - \mu_u}{\sqrt{2}} = -\sqrt{2}\mu_u \propto \mu_u. \quad (25)$$

That is, in the decay of a vector meson to a scalar meson plus photon, $\langle q\bar{q} | \mu | q\bar{q} \rangle \propto \mu_q$. Then,

$$M_{\rho^0 \rightarrow \pi^0 \gamma} \propto \frac{\mu_u + \mu_d}{2} \approx \frac{e}{6m_u}, \quad (26)$$

$$M_{\omega \rightarrow \pi^0 \gamma} = \langle (u\bar{u} - d\bar{d})/\sqrt{2} | \mu | (u\bar{u} + d\bar{d})/\sqrt{2} \rangle \propto \frac{\mu_u - \mu_d}{2} \approx \frac{e}{2m_u}, \quad (27)$$

supposing that $m_u \approx m_d$ and that $\mu_q = Q_q/2m_q$. Recall from p. 193, Lecture 11 of the Notes that a two-body decay rate is proportional to $|M|^2 P_f/m_i^2$. To determine P_f we note the 4-vector relation $p_\pi = p_i - p_\gamma$, whose square is

$$m_\pi^2 = m_i^2 + m_\gamma^2 - 2p_i \cdot p_\gamma = m_i^2 - 2m_i E_\gamma = m_i^2 - 2m_i P_f, \quad (28)$$

$$P_f = \frac{m_i}{2} \left(1 - \frac{m_\pi^2}{m_i^2} \right) \approx \frac{m_i}{2}. \quad (29)$$

Since m_ρ is very close to m_ω , we predict that

$$\Gamma_{\omega^0 \rightarrow \pi^0 \gamma} = 9\Gamma_{\rho^0 \rightarrow \pi^0 \gamma} \propto \frac{e^2}{8m_u^2 m_\rho}, \quad (30)$$

using $m_\rho = 772$ MeV and $m_\phi = 1020$ MeV.

The matrix elements for the decays ρ^0 , ω , $\phi \rightarrow \eta\gamma$ have the form

$$M_{\rho^0 \rightarrow \eta\gamma} = \langle (2s\bar{s} - u\bar{u} - d\bar{d})/\sqrt{6} | \mu | (u\bar{u} - d\bar{d})/\sqrt{2} \rangle \propto \frac{-\mu_u + \mu_d}{\sqrt{12}} \approx -\frac{\sqrt{3}e}{12m_u}, \quad (31)$$

$$M_{\omega \rightarrow \eta\gamma} = \langle (2s\bar{s} - u\bar{u} - d\bar{d})/\sqrt{6} | \mu | (u\bar{u} + d\bar{d})/\sqrt{2} \rangle \propto \frac{-\mu_u - \mu_d}{2} \approx -\frac{\sqrt{3}e}{36m_u}, \quad (32)$$

$$M_{\phi \rightarrow \eta\gamma} = \langle (2s\bar{s} - u\bar{u} - d\bar{d})/\sqrt{6} | \mu | s\bar{s} \rangle \propto \frac{2\mu_s}{\sqrt{6}} \approx -\frac{\sqrt{6}e}{18m_u}, \quad (33)$$

supposing that $m_s \approx 3m_u/2$. For the decays ρ^0 , $\omega \rightarrow \eta\gamma$, eq (29) indicates that $P_f = m_\rho(1 - m_\eta^2/m_\rho^2)/2 \approx m_\rho/4$, while for $\phi \rightarrow \eta\gamma$ we have that $P_f = m_\phi(1 - m_\eta^2/m_\phi^2)/2 \approx 3m_\phi/8$, so the decay rates are predicted to be

$$\Gamma_{\rho^0 \rightarrow \eta\gamma} \propto \frac{e^2}{192m_u^2m_\rho} = \frac{3}{8}\Gamma_{\rho^0 \rightarrow \pi^0\gamma}, \quad (34)$$

$$\Gamma_{\omega \rightarrow \eta\gamma} \propto \frac{e^2}{1728m_u^2m_\rho} = \frac{1}{24}\Gamma_{\rho^0 \rightarrow \pi^0\gamma}, \quad (35)$$

$$\Gamma_{\phi \rightarrow \eta\gamma} \propto \frac{e^2}{144m_u^2m_\phi} \approx \frac{e^2}{192m_u^2m_\rho} = \frac{3}{8}\Gamma_{\rho^0 \rightarrow \pi^0\gamma}, \quad (36)$$

using $m_\rho \approx 3m_\phi/4$.

Referring to <http://pdg.lbl.gov/2013/tables/rpp2013-tab-mesons-light.pdf>, the data are

$$\begin{aligned} \Gamma_{\rho^0} = 149 \pm 0.8 \text{ MeV}, \quad & \frac{\Gamma_{\rho^0 \rightarrow \pi^0\gamma}}{\Gamma_\rho} = 6.0 \pm 0.8 \times 10^{-4}, \quad \Rightarrow \Gamma_{\rho^0 \rightarrow \pi^0\gamma} = 90 \pm 12 \text{ keV}, \\ & \frac{\Gamma_{\rho^0 \rightarrow \eta\gamma}}{\Gamma_\rho} = 3.0 \pm 0.2 \times 10^{-4}, \quad \Rightarrow \Gamma_{\rho^0 \rightarrow \eta\gamma} = 45 \pm 6 \text{ keV}, \\ \Gamma_\omega = 8.49 \pm 0.08 \text{ MeV}, \quad & \frac{\Gamma_{\omega \rightarrow \pi^0\gamma}}{\Gamma_\omega} = 8.3 \pm 0.3 \times 10^{-2}, \quad \Rightarrow \Gamma_{\omega \rightarrow \pi^0\gamma} = 705 \pm 30 \text{ keV}, \\ & \frac{\Gamma_{\omega \rightarrow \eta\gamma}}{\Gamma_\omega} = 4.6 \pm 0.4 \times 10^{-4}, \quad \Rightarrow \Gamma_{\omega \rightarrow \eta\gamma} = 3.9 \pm 0.4 \text{ keV}, \\ \Gamma_\phi = 4.26 \pm 0.4 \text{ MeV}, \quad & \frac{\Gamma_{\phi \rightarrow \pi^0\gamma}}{\Gamma_\phi} = 1.27 \pm 0.06 \times 10^{-3}, \quad \Rightarrow \Gamma_{\phi \rightarrow \pi^0\gamma} = 5.4 \pm 0.3 \text{ keV}, \\ & \frac{\Gamma_{\phi \rightarrow \eta\gamma}}{\Gamma_\phi} = 1.31 \pm 0.03 \times 10^{-2}, \quad \Rightarrow \Gamma_{\phi \rightarrow \eta\gamma} = 56 \pm 1.3 \text{ keV}, \\ \Gamma_J = 92.9 \pm 2.8 \text{ keV}, \quad & \frac{\Gamma_{J \rightarrow \pi^0\gamma}}{\Gamma_\phi} = 3.5 \pm 0.3 \times 10^{-5}, \quad \Rightarrow \Gamma_{J \rightarrow \pi^0\gamma} = 0.33 \pm 0.03 \text{ keV}. \end{aligned}$$

The data are in fairly good agreement with the predictions, although the nonzero value $\Gamma_{\phi \rightarrow \pi^0\gamma}$ suggests that the ϕ is not quite a pure $s\bar{s}$ state.

- The reaction $e^+e^- \rightarrow \psi' \rightarrow \chi\gamma$ proceeds, to the first approximation, via e^+e^- annihilation into a single photon, which materializes as the $\psi'(3685)$ that then decays to $\chi(3415)\gamma$. The electron/positron beam energy of $3685/2$ MeV is large compared to the electron mass, so the reaction takes place in the high-energy limit where the annihilation occurs only for $S_z = \pm 1$ (transverse photons).

Since the $\chi(3415)$ has zero spin, the final state $\chi(3415)\gamma$ (with a real photon) can only have $S_{z'} = \pm 1$ along the axis of the final state momenta at angle θ to the beam (z)

axis. The angular dependence of the matrix element in the center-of-mass frame is then the sum of the projections of $S_z = \pm 1$ onto $S_{z'} = \pm 1$, as described by the spin-1 rotation matrix (the d functions summarized at

<http://pdg.lbl.gov/2013/reviews/rpp2012-rev-clebsch-gordan-coefs.pdf>). That is,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &\propto (d_{1,1}^1)^2 + (d_{1,-1}^1)^2 + (d_{-1,1}^1)^2 + (d_{-1,-1}^1)^2 \\ &= \left(\frac{1+\cos\theta}{2}\right)^2 + \left(\frac{1-\cos\theta}{2}\right)^2 + \left(\frac{1-\cos\theta}{2}\right)^2 + \left(\frac{1+\cos\theta}{2}\right)^2 \\ &= 1 + \cos^2\theta. \end{aligned} \quad (37)$$

This result is, of course, the same as that for the reaction $e^+e^- \rightarrow \mu^+\mu^-$.

4. The square of the matrix element for the 4-particle vertex $\mu\nu_\mu e\nu_e$ of unpolarized particles in the Fermi theory of the weak interaction is, in terms of the particle 4-vectors p ,

$$|M|^2 = 32G_F^2(p_\mu \cdot p_{\nu_e})(p_e \cdot p_{\nu_\mu}), \quad (38)$$

where G_F is Fermi's constant, and the average over initial spins and sum over final spins is the same for all variants of the vertex.

$\sigma_{\nu_\mu+e^- \rightarrow \mu^-+\nu_e}$

Since the center of mass energy must be at least $\sqrt{s} = m_\mu$ for this reaction to proceed, it suffices to approximate the electron as relativistic in the center of mass frame. Then, $E_e^* \approx E_{\nu_\mu}^* \approx \sqrt{s}/2 = E_i^*$ in this frame, and we can write the 4-vectors as

$$\begin{aligned} p_{\nu_\mu} &\approx E_i^*(1, 0, 0, 1), & p_e &\approx E_i^*(1, 0, 0, -1), \\ p_\mu &= (E_\mu^*, P_f^* \sin\theta^*, 0, P_f^* \cos\theta^*), & p_{\nu_e} &= P_f^*(1, -\sin\theta^*, 0, -\cos\theta^*), \end{aligned} \quad (39)$$

approximating the neutrinos as massless.

Conservation of energy implies that $2E_i^* = E_\mu^* + E_{\nu_e}^* = E_\nu^* + P_f^*$, such that

$$E_\nu^* = 2E_i^* - P_f^* = \sqrt{P_f^{*2} + m_\mu^2}, \quad \Rightarrow \quad P_f^* = E_i^* \left(1 - \frac{m_\mu^2}{4E_i^{*2}}\right) = E_i^* \left(1 - \frac{m_\mu^2}{s}\right). \quad (40)$$

Hence,

$$\begin{aligned} p_e \cdot p_{\nu_\mu} &\approx 2E_i^{*2}, \\ p_\mu \cdot p_{\nu_e} &= P_f^*(E_\mu^* + P_f^* \sin^2\theta^* + P_f^* \cos^2\theta^*) = P_f^*(E_\mu^* + P_f^*) = 2E_i^*P_f^*, \end{aligned} \quad (41)$$

using energy conservation in the last step. The square of the matrix element, eq. (38), is independent of angle,

$$|M|^2 \approx 128G_F^2 E_i^{*3} P_f^*, \quad (42)$$

so the total cross section is, noting that in the center of mass frame $P_i^* \approx E_i^*$ (and that the cross section is isotropic),

$$\begin{aligned}\sigma_{\nu_\mu + e^- \rightarrow \mu^- + \nu_e} &= \int \frac{d\sigma}{d\Omega^*} d\Omega^* = 4\pi \frac{d\sigma}{d\Omega^*} = 4\pi \frac{|M|^2 P_f^*}{64\pi^2 s P_i^*} \approx \frac{4\pi \cdot 128 G_F^2 E_i^{*3} P_f^{*2}}{64\pi^2 (2E_i^*)^2 E_i^*} \\ &= \frac{2G_F^2 E_i^{*2}}{\pi} \left(1 - \frac{m_\mu^2}{4E_i^*}\right)^2 = \frac{G_F^2 s}{2\pi} \left(1 - \frac{m_\mu^2}{s}\right)^2.\end{aligned}\quad (43)$$

using $s = (p_e + p_{\nu_\mu})^2 = 4E_i^{*2}$ in the last step.

In the lab frame, $p_e = (m_e, 0, 0, 0)$ and $p_{\nu_\mu} = E(1, 0, 0, 1)$, so $s = (p_e + p_{\nu_\mu})^2 = 2m_e E + m_e^2 \approx 2m_e E$, so the cross section rises linearly with the energy E of the neutrino beam,

$$\sigma_{\nu_\mu + e^- \rightarrow \mu^- + \nu_e} \approx \frac{G_F^2 m_e E}{\pi} \left(1 - \frac{m_\mu^2}{2m_e E}\right)^2. \quad (44)$$

Can this cross section really grow infinitely large at high energy?

5. $\sigma_{e^+ e^- \rightarrow \nu_e \bar{\nu}_e}$

In the Fermi theory this process has the squared matrix element (38) on replacing the symbol μ by e .

In the center of mass frame we define the 4-vectors to be (for massless neutrinos)

$$\begin{aligned}p_{e^+} &= (E^*, 0, 0, P^*), & p_{e^-} &= (E^*, 0, 0, -P^*), \\ p_{\nu_e} &= E_\nu^*(1, \sin \theta^*, 0, \cos \theta^*), & p_{\bar{\nu}_e} &= E_\nu^*(1, -\sin \theta^*, 0, -\cos \theta^*),\end{aligned}\quad (45)$$

Conservation of energy can be expressed as $s = (p_{e^+} + p_{e^-})^2 = 4E^{*2} = (p_{\nu_e} + p_{\bar{\nu}_e})^2 = 4E_\nu^{*2}$ such that $E_\nu^* = E^*$. Of course, $P^* = \sqrt{E^{*2} - m_e^2}$.

Then,

$$\begin{aligned}p_{e^-} \cdot p_{\bar{\nu}_e} &= E^*(E^* - P^* \cos \theta^*), \\ p_{e^+} \cdot p_{\nu_e} &= E^*(E^* - P^* \cos \theta^*),\end{aligned}\quad (46)$$

$$|M|^2 = 32G_F^2 E^{*2} (E^{*2} - 4E^* P^* \cos \theta^* + P^{*2} \cos^2 \theta^*), \quad (47)$$

and the cross section in the center-of-mass frame (where $P_f = E^*$ and $P_i = P^*$) is

$$\begin{aligned}\sigma_{e^+ e^- \rightarrow \nu_e \bar{\nu}_e} &= \int \frac{d\sigma}{d\Omega^*} d\Omega^* = 2\pi \int_{-1}^1 \frac{|M|^2 P_f}{64\pi^2 s P_i} d\cos \theta^* \\ &= \frac{2\pi \cdot 32G_F^2 E^{*3} (2E^{*2} + 2P^{*2}/3)}{64\pi^2 (2E^*)^2 P^*} \\ &= \frac{G_F^2 E^* (4E^{*2} - m_e^2)}{6\pi P^*} = \frac{G_F^2 s}{6\pi} \frac{1 - m_e^2/s}{\sqrt{1 - 4m_e^2/s}}.\end{aligned}\quad (48)$$

The cross section diverges near threshold, where the electron and positron would form a positronium “atom” rather than scatter.

This result was first obtained by H.-Y. Chiu and P. Morrison, *Neutrino Emission from Black-Body Radiation at High Stellar Temperatures*, *Phys. Rev. Lett.* **5**, 573 (1960), http://physics.princeton.edu/~mcdonald/examples/EP/chiu_pr1_5_573_60.pdf, and in slightly more detail in H.Y. Chiu, *Annihilation Process of Neutrino Production in Stars*, *Phys. Rev.* **123**, 1040 (1961),

http://physics.princeton.edu/~mcdonald/examples/EP/chiu_pr_123_1040_61.pdf.

The cross section as modified by the Weinberg-Salam model was first discussed by D.A. Dicus, *Stellar Energy-Loss Rates in a Convergent Theory of Weak and Electromagnetic Interactions*, *Phys. Rev. D* **6**, 941 (1972),

http://physics.princeton.edu/~mcdonald/examples/EP/dicus_prd_6_941_72.pdf.

6. Decay of positronium, $e^+e^- \rightarrow \nu_e\bar{\nu}_e$

If a positronium atom at rest decays to $\nu_e\bar{\nu}_e$, the neutrinos are collinear, and have total spin 1 along the decay axis. Hence, the 1S_0 (para-positronium) state cannot decay to two neutrinos, but the 3S_1 (ortho-positronium) state can.

The decay rate $\Gamma_{e^+e^- \rightarrow \nu_e\bar{\nu}_e}$ can be written in terms of the scattering cross section as

$$\Gamma_{e^+e^- \rightarrow \nu_e\bar{\nu}_e} = N\sigma_{e^+e^- \rightarrow \nu_e\bar{\nu}_e} v_{\text{rel}}, \quad (49)$$

where N is the number of “scatters” (in which the electron and positron meet at the center of the positronium atom) per unit volume, which can be taken as equal to the probability $|\psi(0)|^2$ of the atomic wave function at the origin,

$$\psi(S \text{ wave}, r) = \frac{e^{-r/a_0}}{\sqrt{\pi}a_0^{3/2}}, \quad \text{where} \quad a_0 = \frac{1}{\alpha m_{\text{reduced}}} = \frac{2}{\alpha m_e}, \quad (50)$$

and the relative velocity v_{rel} can be written as

$$v_{\text{rel}} = 2v_e = \frac{2P^*}{E^*}, \quad (51)$$

where $E^* \approx m_e$. Thus, recalling eq. (48),

$$\Gamma_{e^+e^- \rightarrow \nu_e\bar{\nu}_e} = \frac{\alpha^3 m_e^3 G_F^2 E^* (4E^{*2} - m_e^2)}{8\pi \cdot 6\pi P^*} \frac{2P^*}{E^*} \approx \frac{\alpha^3 G_F^2 m_e^5}{8\pi^2}. \quad (52)$$

The decay rate $\mu^+e^- \rightarrow \nu_e\bar{\nu}_\mu$ was first computed by P.-J. Li, Z.-Q. Tan and C.-E. Wu, *Weak decays of polarised muonium and polarised pionium*, *J. Phys. G: Nucl. Phys.* **14**, 525 (1988), http://physics.princeton.edu/~mcdonald/examples/EP/li_jpg_14_525_88.pdf.