

INTRODUCTION TO WAVES (LECTURES 11-14 CORRESPOND TO BECKER CHAPTER DII)

WAVES IN FREE SPACE

IN THE REMAINDER OF THE COURSE WE EXPLORE SOME OF THE RICHES OF TIME-DEPENDENT ELECTRO-MAGNETIC PHENOMENA. THE TERM 'ELECTRO-MAGNETIC' REMINDS US THAT MAXWELL'S EQUATIONS FOR TIME-DEPENDENT FIELDS LINK THE ELECTRICAL AND THE MAGNETIC FIELDS, AND HENCE FORTH WE WILL ALWAYS CONSIDER THE TWO TOGETHER.

WE DEFER FOR A WHILE THE QUESTION OF HOW THE FIELDS ARE RELATED TO MOTIONS OF THE CHARGES WHICH CAUSE THEM. FOLLOWING FARADAY AND MAXWELL WE CONSIDER THE FIELDS IN THEIR OWN RIGHT, FREE FROM THEIR SOURCES. INITIALLY WE CONSIDER THE FIELDS IN EMPTY SPACE...

THEN MAXWELL TELLS US

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

HENCE

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c} \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$$

SO

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{LIKEWISE} \quad \nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

WE RECOGNIZE THESE AS WAVE EQUATIONS IN 3 DIMENSIONS.

THAT IS, ANY COMPONENT OF \vec{E} WHICH OBEYS

$$E_i = f(x-ct) + g(x+ct)$$

WILL SATISFY THE ABOVE EQUATION.

OF COURSE C IS WAVE VELOCITY, ALTHOUGH C WAS

ORIGINALLY INTRODUCED AS A DIMENSIONAL CONSTANT IN THE EXPRESSION FOR MAGNETOSTATIC FORCES.

SINCE OUR WAVE EQUATION HAS 3 SPATIAL DIMENSIONS OTHER KINDS OF WAVE SOLUTIONS ARE POSSIBLE:

$f(y-ct)$ OR $f(z-ct)$ AND EVEN $f\left(\frac{\sqrt{x^2+y^2+z^2}-ct}{\sqrt{x^2+y^2+z^2}}\right)$

ARE ALL SOLUTIONS.

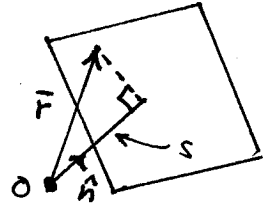
PLANE WAVES

AMONG THE LARGE VARIETY OF WAVE SOLUTIONS IT IS USEFUL TO FOCUS OUR ATTENTION ON A SPECIAL CLASS - THE PLANE WAVES.

IN THESE SOLUTIONS \vec{E} AND \vec{B} TAKE ON CONSTANT VALUES OVER A PLANE - BUT MAY TAKE ON A DIFFERENT VALUE ON ANOTHER PARALLEL PLANE.

SOME GEOMETRY: IF S IS THE PERPENDICULAR DISTANCE FROM THE ORIGIN TO THE PLANE, THEN ANY POINT IN THE PLANE OBEYS

$$\vec{r} \cdot \hat{n} = S,$$



\hat{n} = UNIT VECTOR ALONG THE DIRECTION S .

$$\begin{aligned} \text{THEN } \vec{E}(\vec{r}, t) &= \vec{E}(S - ct) = \vec{E}(\vec{r} \cdot \hat{n} - ct), \\ \vec{B}(\vec{r}, t) &= \vec{B}(\vec{r} \cdot \hat{n} - ct). \end{aligned}$$

ARE THE PLANE WAVE SOLUTIONS. A PLANE OF CONSTANT FIELD MOVES IN THE \hat{n} DIRECTION WITH VELOCITY c .

A VERY IMPORTANT PROPERTY OF THE PLANE WAVES IS THAT THE FIELDS ARE TRANSVERSE TO THE DIRECTION OF MOTION, AND FURTHER, THAT \vec{E} IS \perp TO \vec{B} .

WE DEMONSTRATE THIS SUPPOSING $\hat{n} = \hat{z}$.

FIRST CONSIDER $\vec{E}(\vec{r}, t)$. THE IDEA OF THE PLANE WAVE IS THAT \vec{E} IS THEN INDEPENDENT OF x AND y .

$$\text{NOW } \vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial E_z}{\partial z} = 0,$$

$\Rightarrow E_z$ IS INDEPENDENT OF z !

$$\text{ALSO } \frac{\partial E_z}{\partial t} = c(\vec{\nabla} \times \vec{B})_z = c\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = 0,$$

CONVERSELY, A WAVE THAT MOVES IN z BUT VARIES IN x & y (EX: A LASER BEAM) CANNOT BE PURELY TRANSVERSE TO z .

SO E_z IS ALSO INDEPENDENT OF TIME.

THUS E_z DOES NOT REALLY PARTICIPATE IN PLANE WAVE MOTION. SIMILARLY FOR B_z .

ONLY TRANSVERSE FIELDS CAN SHOW TIME-DEPENDENT PLANE WAVE MOTION.

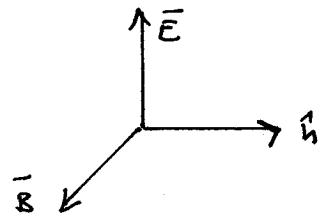
$$\text{FURTHERMORE: } \frac{1}{c} \frac{\partial E_x}{\partial t} = (\vec{\nabla} \times \vec{B})_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{\partial B_y}{\partial z}$$

$$\text{SO IF } E_x = f(z - ct), \quad \frac{1}{c} (-cf') = -\frac{\partial B_y}{\partial z} \Rightarrow B_y = f = E_x$$

LIKEWISE $\frac{1}{c} \frac{\partial E_y}{\partial t} = (\nabla \times \vec{B})_y \Rightarrow -B_x = E_y$

THIS MAY BE SUMMARISED AS $\vec{E} = \vec{B} \times \hat{u}$.

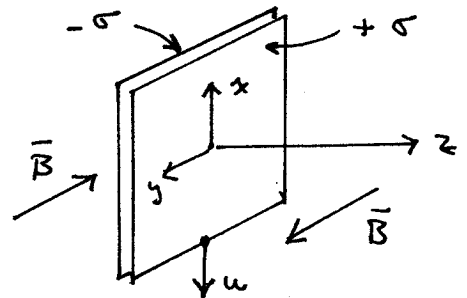
SO, OF COURSE, $\vec{E} \cdot \vec{B} = 0$ AND $|\vec{E}| = |\vec{B}|$



A SKETCH OF HOW IT WORKS

WE GIVE AN IDEALIZED EXAMPLE OF HOW WE MIGHT GENERATE A PLANE WAVE.

SUPPOSE WE HAVE TWO INFINITE SHEETS OF CHARGE LYING IN THE PLANE $z=0$. ONE HAS CHARGE DENSITY $+\sigma$, THE OTHER $-\sigma$. AT $t=0$, WE BEGIN MOVING THE $+\sigma$ SHEET IN THE $-x$ DIRECTION WITH VELOCITY u .



THEN THE MOVING CHARGE CAUSES A MAGNETIC FIELD. BY AMPERE'S LAW WE CONCLUDE THAT $\vec{B} = \pm B \hat{y}$

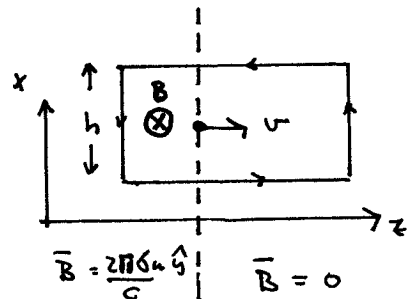
WITH $2lB = \frac{4\pi}{c} \cdot I$ CROSSING LENGTH $\Delta y = l = \frac{4\pi}{c} \sigma l u$

OR $B = \frac{2\pi}{c} \sigma u$

NOW THIS REALLY MAKES SENSE ONLY IF THE CURRENT HAS BEEN FLOWING FOREVER. BUT WE STARTED THE CURRENT AT $t=0$. MAYBE IT TAKES A FINITE AMOUNT OF TIME BEFORE THE AMPERE'S LAW FIELD IS OBSERVED AT $z \neq 0$. WE IMAGINE THE REGION OF NON-ZERO \vec{B} FIELD MOVES OUTWARDS WITH SOME VELOCITY $\vec{v} = v \hat{z}$ (OR $\vec{v} = -v \hat{z}$ FOR $z < 0$).

BUT FARADAY'S LAW TELLS US THAT A CHANGING \vec{B} FIELD INDUCES AN ELECTRIC FIELD.

CONSIDER A LOOP IN THE $x-z$ PLANE AS SHOWN. THE TRANSITION FROM $\vec{B} = 0$ TO $\vec{B} > 0$ IS MOVING ACROSS THE LOOP WITH VELOCITY v .



FARADAY: $\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{d}{dt} \int \vec{B} \cdot d\vec{S} = -\frac{B_y v h}{c}$

AGAIN, WE EXPECT $\vec{E} = 0$ ON THE LOOP OUTSIDE THE REGION WHERE $\vec{B} \neq 0$ - TO MAINTAIN CAUSE AND EFFECT.

SO $\oint \vec{E} \cdot d\vec{l} = -E_x h \Rightarrow E_x = \frac{v}{c} B_y$

THIS ARGUMENT HOLDS ANYWHERE INSIDE THE REGION OF NON-VANISHING \vec{B} . HENCE \vec{E} IS CONSTANT THERE - AND THE BOUNDARY OF THE REGION OF NON-VANISHING \vec{E} ALSO MOVES OUTWARDS AT VELOCITY v .

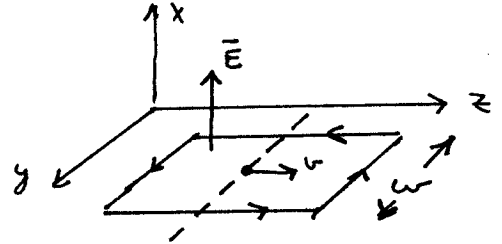
WE CAN PLAY THE GAME AGAIN - NOW WITH A LOOP IN THE $y-z$ PLANE. MAXWELL'S 4TH EQUATION TELLS US

$$\oint \vec{B} \cdot d\vec{l} = \frac{1}{c} \frac{d}{dt} \int \vec{E} \cdot d\vec{S} = \frac{v}{c} E_x w$$

$$\parallel$$

$$B_y \cdot w$$

$$\text{so } B_y = \frac{v}{c} E_x = \left(\frac{v}{c}\right) B_y$$



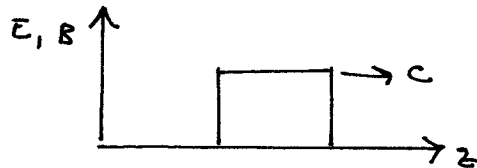
THIS CAN ONLY BE POSSIBLE IF $v = c =$ VELOCITY OF THE 'WAVE FRONT'!

THEN \vec{E} IS \perp TO \vec{B} AND $|\vec{E}| = |\vec{B}|$ ALSO!

SUPPOSE AT SOMETIME $t > 0$ WE STOP THE MOTION OF THE CHARGED SHEET. DO THE FIELDS JUST VANISH?

NO! OF COURSE, VERY NEAR THE SHEET THE FIELDS DO GO AWAY. BUT IT TAKES A FINITE AMOUNT OF TIME FOR THE MESSAGE TO PROPAGATE THAT THE CHARGES HAVE STOPPED MOVING. WE EXPECT A SORT OF 'WAVE FRONT' WHICH MOVES OUTWARDS AT VELOCITY c . BUT THIS LEAVES A REGION OF NON-VANISHING \vec{E} AND \vec{B} WHICH IS NO LONGER CONNECTED TO THE CHARGES.

THIS REGION SEEMS TO HAVE A LIFE OF ITS OWN - CONTINUOUSLY CREATING NEW FIELD AT ITS LEADING EDGE, AND DESTROYING FIELD AT ITS TRAILING EDGE.



ALL DESCRIBED BY MAXWELL'S EQUATIONS!

HARMONIC PLANE WAVES

WAVES WITH OSCILLATORY TIME DEPENDENCE ARE CLEARLY OF GREAT INTEREST: $\vec{E} = \vec{E}(\vec{r}) e^{-i\omega t}$ [$\vec{E}(\vec{r})$ IS COMPLEX]

FOR A PLANE WAVE $\vec{E} = \vec{E}(\vec{r} \cdot \hat{n} - ct)$ TO BE OSCILLATORY WE WRITE

$$c = \frac{\omega}{k} \quad k = \frac{2\pi}{\lambda} \quad \vec{k} = k \hat{n}$$

$$\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

WITH $\vec{E}_0 =$ CONSTANT VECTOR OBEYING $\vec{E}_0 \cdot \vec{k} = 0$
 [\vec{E}_0 IS COMPLEX IN GENERAL]

LIKEWISE $\vec{B} = \vec{B}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

AND $\vec{B}_0 = \hat{n} \times \vec{E}_0 = \hat{k} \times \vec{E}_0$

WE OF COURSE MEAN $\vec{E} = \text{Re} \left\{ \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right\}$ ETC.

THIS NOTATION ALLOWS THE POSSIBILITY THAT \vec{E}_0 IS COMPLEX, WHICH IS QUITE USEFUL, TWO PERHAPS INITIALLY CONFUSING.

FOR EXAMPLE, SUPPOSE $\hat{k} = \hat{z}$

THEN IF $\vec{E}_0 = E_{0x} e^{i\theta_x} \hat{x} + E_{0y} e^{i\theta_y} \hat{y}$

WE REALY MEAN $\vec{E} = E_{0x} \cos(kz - \omega t + \theta_x) \hat{x} + E_{0y} \cos(kz - \omega t + \theta_y) \hat{y}$

A BOUNDED SOURCE CANNOT EMIT A UNIPOLAR PULSE

THE PEDAGOGIC EXAMPLE ON PP. 124-125 PROVIDES AN EXAMPLE OF HOW A ONE-DIMENSIONAL WAVE CAN BE UNIPOLAR. WE

FOUND $E_x, B_y(z-ct)$ WHERE $E_x \geq 0, B_y \geq 0$ EVERYWHERE.

HOWEVER, THIS IS NOT POSSIBLE FOR THREE-DIMENSIONAL WAVES, WITH THE REALISTIC CONSTRAINT THAT THE SOURCE OF THE WAVES LIES WITHIN A BOUNDED VOLUME.

FIRST, WE DEDUCE THAT FAR FROM THE SOURCE, TAKEN TO BE NEAR THE ORIGIN, WAVE FIELDS $\vec{E} \neq \vec{B}$ FALL OFF AS $1/r$. THIS FOLLOWS FROM ENERGY CONSERVATION:

$$\int_{\text{SPHERE OF RADIUS } r} \vec{S} \cdot d\text{AREA} \rightarrow \frac{E \cdot B(r)}{4\pi c} \cdot 4\pi r^2 \rightarrow \text{CONSTANT FOR LARGE } r$$

SINCE $|\vec{E}| = |\vec{B}|$ FOR WAVE FIELDS (FAR FROM THE SOURCE \Rightarrow ESSENTIALLY LIKE PLANE WAVES IN ANY LOCAL REGION), WE MUST HAVE

$$E_{\text{WAVE}} = B_{\text{WAVE}} \propto \frac{1}{r} \quad (\text{LARGE } r)$$

THIS CONTRASTS WITH STATIC FIELDS, WHOSE LEADING BEHAVIOR FAR FROM A BOUNDED SOURCE IS (RECALL THE MULTIPOLE EXPANSIONS)

$$E_{\text{STATIC}} \sim \frac{1}{r^2}, \quad B_{\text{STATIC}} \sim \frac{1}{r^3}$$

SUPPOSE, NOW, THAT A WAVE FIELD WERE UNIPOLAR. THIS MEANS

$$\int E(\vec{r}, t) dt \neq 0 \quad \text{AT POINTS } \vec{r} \text{ FAR FROM THE SOURCE.}$$

WE COULD MAKE A FOURIER ANALYSIS OF THIS WAVE:

$$E(\vec{r}, \omega) = \int E(\vec{r}, t) e^{i\omega t} dt.$$

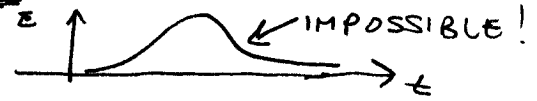
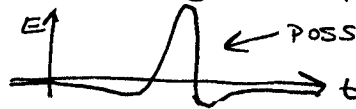
IN PARTICULAR, WE WOULD OBTAIN A NONZERO VALUE AT ZERO FREQUENCY:

$$E(\vec{r}, \omega=0) = \int E(\vec{r}, t) dt \neq 0.$$

BUT THE ZERO FREQUENCY "WAVE" IS ACTUALLY A STATIC SOLUTION TO MAXWELL'S EQUATIONS. HENCE, IT CANNOT FALL OFF AS $1/r$, AS WAS ASSUMED.

BY CONTRADICTION, WE CONCLUDE THAT ALL 3-D WAVES OBEY

$$\int E(\vec{r}, t) dt = 0.$$



POLARIZATION

THE BEHAVIOR SYMBOLIZED BY THE COMPLEX VECTOR \vec{E}_0 IS CALLED THE POLARIZATION OF THE WAVE.

CLEARLY THE SIMPLEST CASE IS $\vec{E}_0 = E_0 \hat{n}$ WITH E_0 REAL AND $\hat{n} \perp$ TO \hat{k} . THIS IS CALLED LINEAR POLARIZATION. FOR WAVES IN A GIVEN DIRECTION THERE ARE 2 INDEPENDENT POSSIBILITIES FOR LINEARLY POLARIZED WAVES. BUT LONGITUDINALLY POLARIZED WAVES ARE NOT POSSIBLE. (THE STATIC FIELD OF A POINT CHARGE MAY BE SAID TO HAVE LONGITUDINAL ELECTRIC POLARIZATION. $\vec{B} = 0$ THEN OF COURSE).

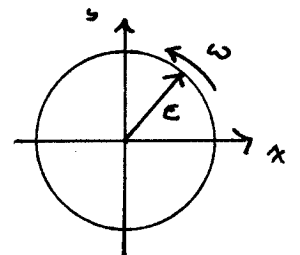
NEXT IN ORDER OF COMPLEXITY ARE CIRCULARLY POLARIZED WAVES.

$$\vec{E}_0 = E_0 (\hat{x} + e^{\pm i\frac{\pi}{2}} \hat{y}) = E_0 (\hat{x} \pm i \hat{y})$$

$$\text{SO } \vec{E} = E_0 (\cos(kz - \omega t) \hat{x} \mp \sin(kz - \omega t) \hat{y})$$

$$\text{AT A FIXED POINT SUCH AS } z=0, \vec{E}|_{z=0} = E_0 (\hat{x} \cos \omega t \pm \hat{y} \sin \omega t)$$

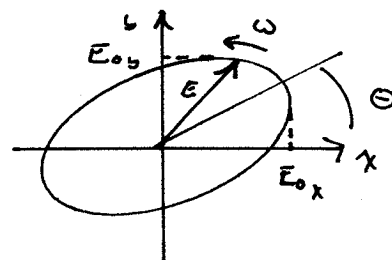
AND \vec{E} APPEARS TO ROTATE IN THE X-Y PLANE AT ANGULAR VELOCITY ω . $|\vec{E}| = \text{CONST.}$



THE GENERAL CASE IS ELLIPTICAL POLARIZATION

$$\vec{E}_0 = E_{0x} \hat{x} + E_{0y} e^{i\theta} \hat{y}$$

AT $z=0$, \vec{E} BEHAVES LIKE



TRICKY TERMINOLOGY: RIGHT-HANDED CIRCULAR POLARIZATION \Rightarrow

\vec{E} VECTOR SWEEPS OUT A RIGHT-HANDED SPIRAL IN SPACE.

BUT AT A FIXED PLANE \perp TO \hat{n} , \vec{E} MOVES CLOCKWISE WITH TIME

1266

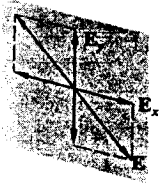
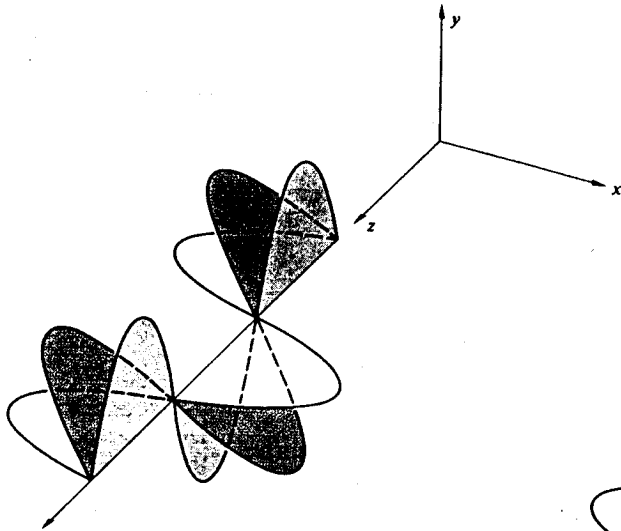


Fig. 8.2 Linear light.

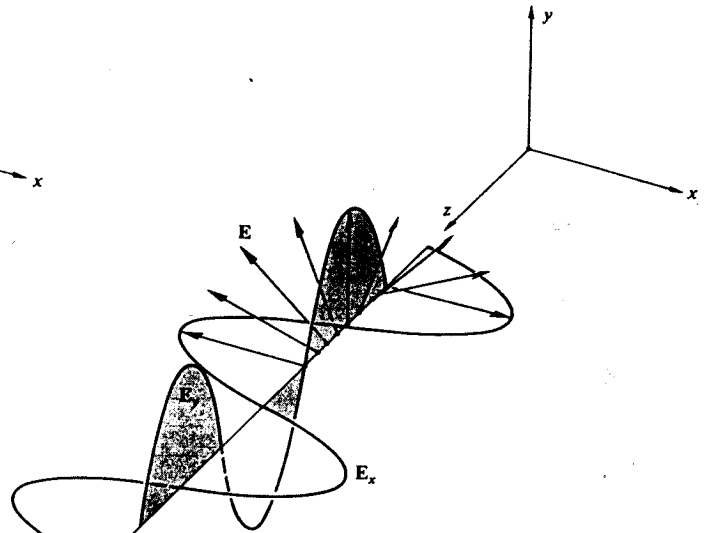


Fig. 8.3 Right circular light.

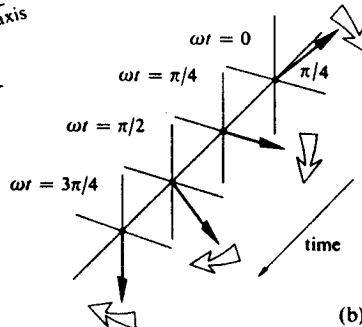
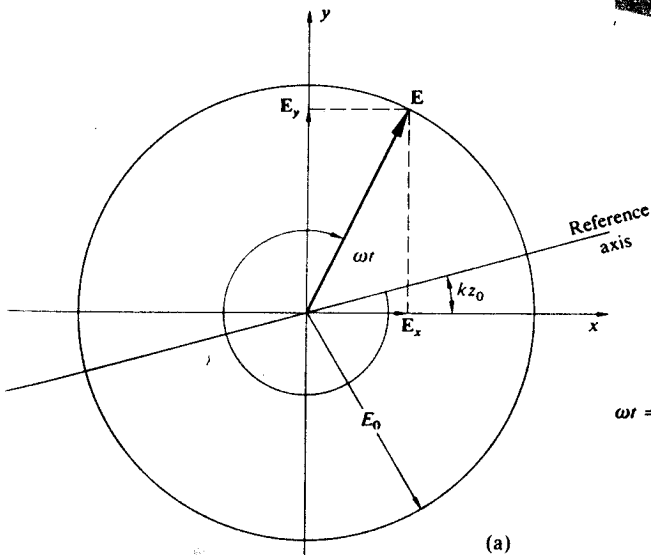


Fig. 8.4 Rotation of the electric vector in a right circular wave. Note that the rotation rate is ω and $kz = \pi/4$.

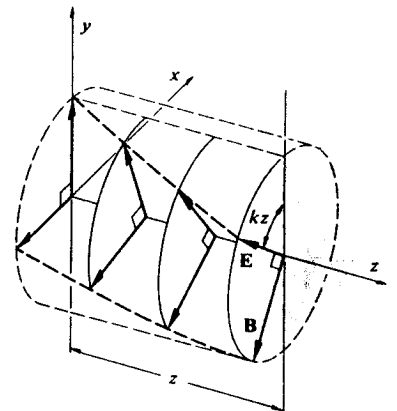


Fig. 8.5 Right circular light.

THE MAGNETIC FIELD \vec{H} (OR \vec{B}) IS ALWAYS \perp TO \vec{E} , AND IN PHASE WITH \vec{E} (SO LONG AS CONDUCTIVITY $\sigma = 0$)

PLANE WAVE CALCULUS

IT IS USEFUL TO NOTE SOME RELATIONS FOR THE DERIVATIVES OF HARMONIC PLANE WAVES!

$$\text{WITH } \bar{E} = E_0 e^{i(\bar{k} \cdot \bar{r} - \omega t)} \quad \frac{\partial \bar{E}}{\partial x_i} = i k_i \bar{E} \quad \text{ETC}$$

SO THE OPERATOR $\bar{\nabla}$ CAN BE REPLACED BY $i\bar{k}$

$$\bar{\nabla} \cdot \bar{E} = 0 \Rightarrow i\bar{k} \cdot \bar{E} = 0 \Rightarrow \bar{E} \perp \text{TO } \bar{k}!$$

$$\text{LIKEWISE } \frac{\partial \bar{E}}{\partial t} = -i\omega \bar{E} \quad \text{ETC}$$

$$\text{SO } \bar{\nabla} \times \bar{E} = -\frac{1}{c} \frac{\partial \bar{B}}{\partial t} \Rightarrow i\bar{k} \times \bar{E} = \frac{i\omega}{c} \bar{B} = i k \bar{B}$$

$$\text{AND } \bar{B} = \hat{k} \times \bar{E}.$$

$$\nabla^2 \bar{E} - \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} = 0 \Rightarrow -k^2 \bar{E} + \frac{\omega^2}{c^2} \bar{E} = 0 \Rightarrow k = \frac{\omega}{c} \quad \text{ETC...}$$

INTEGRALS OF THE PLANE WAVES ARE USEFUL, A LA FOURIER

$$E_x(z-ct) = \int_{-\infty}^{\infty} A(\omega) e^{i(kz - \omega t)} d\omega = \int_{-\infty}^{\infty} A(\omega) e^{-i\omega(t - \frac{z}{c})} d\omega$$

$$\text{WITH } A(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_x(-ct) e^{i\omega t} dt \quad (\text{FROM ABOVE AT } z=0)$$

SO GENERAL WAVES CAN BE BUILT UP OUT OF HARMONIC PLANE WAVES.

ENERGY AND MOMENTUM

IN LECTURE 10, WE FOUND WE COULD DEFINE A FIELD ENERGY DENSITY

$$u = \frac{dU}{dvol} = \frac{1}{8\pi} (\bar{E} \cdot \bar{D} + \bar{B} \cdot \bar{H})$$

$$= \frac{1}{8\pi} (E^2 + B^2) \quad \text{IN FREE SPACE}$$

WHEN DEALING WITH OSCILLATORY WAVES, IT IS CUSTOMARY TO AVERAGE OVER A PERIOD. SO IF $E = E_0 \cos(kz - \omega t)$

$$\langle E^2 \rangle = \frac{E_0^2}{2}$$

$$\text{AND } \langle u \rangle = \frac{1}{16\pi} (E_0^2 + B_0^2) = \frac{E_0^2}{8\pi} \quad \text{NOTING } E_0 = B_0$$

THE ELECTRIC AND MAGNETIC PARTS OF $\langle u \rangle$ ARE EQUAL!

WE REMARK THAT IF $\vec{E} = \text{Re} \left\{ \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right\}$ WITH COMPLEX \vec{E}_0

THEN $\langle E^2 \rangle = \frac{1}{2} \text{Re} (\vec{E} \cdot \vec{E}^*)$ WHICH IS OFTEN USEFUL.

IN LECTURE 10 WE ALSO OBTAINED AN EXPRESSION FOR THE FLOW OF ENERGY, DESCRIBED BY POYNTING'S VECTOR

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} = \frac{c}{4\pi} \vec{E} \times \vec{B} \quad \text{IN FREE SPACE}$$

WITH $\vec{B} = \hat{k} \times \vec{E}$ THIS IS $\vec{S} = \frac{c}{4\pi} E^2 \hat{k}$

AND $\langle S \rangle = \frac{c}{8\pi} \langle E^2 \rangle \hat{k} = \langle u \rangle c \hat{k}$

SO THE ENERGY FLOW IS INDEED THE ENERGY DENSITY TIMES THE VELOCITY OF THAT DENSITY! NOW THE POYNTING VECTOR BEGINS TO MAKE MORE SENSE.

WITH \vec{E} AND \vec{H} POSSIBLY COMPLEX, WE MAY WRITE

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \text{Re} [\vec{E} \times \vec{H}^*]$$

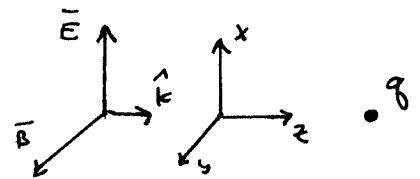
RECALL FROM LECTURE 10 THAT WE MADE AN ADDITIONAL INTERPRETATION OF THE POYNTING VECTOR!

$$\vec{P}_{\text{FIELD}} = \frac{\vec{S}}{c^2} = \text{MOMENTUM DENSITY}$$

FOR OUR PLANE WAVE, WE CAN ALSO WRITE $\langle P_{\text{WAVE}} \rangle = \langle u \rangle c \hat{k}$

EXAMPLE A PLANE WAVE INCIDENT ON A CHARGE

WE CONSIDER A LINEARLY POLARIZED PLANE WAVE. THE ELECTRIC FIELD OF THE WAVE THEN SHAKES THE CHARGE WITH THE OSCILLATORY FORCE



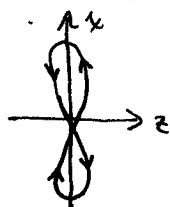
$$F_x = q E_0 \sin \omega t = m \ddot{x} \Rightarrow v_x = \frac{q E_0 \sin \omega t}{m \omega} \quad \& \quad x = -\frac{q E_0}{m \omega^2} \cos \omega t$$

WHICH CAUSES OSCILLATORY MOTION ALONG THE \hat{k} DIRECTION.

THEN THE MAGNETIC FIELD PUSHES ON THE MOVING CHARGE

WITH FORCE $\vec{F} = q \frac{\vec{v}}{c} \times \vec{B} = q \frac{v_x}{c} B_y \hat{z} = \frac{q^2 E_0^2 \sin^2 \omega t \cos \omega t}{m \omega c} = \frac{q^2 E_0^2 \sin 2\omega t}{2 m \omega c} = m \ddot{z}$

$$z = -\frac{q^2 E_0^2}{8 m \omega^3 c} \sin 2\omega t$$



SUBTLETY: IF A PARTICLE AT REST IS OVERTAKEN BY A WAVE, IT PICKS UP A DRIFT VELOCITY v_z

$$\frac{v_z}{c} = \frac{v^2}{2 + v^2} \quad \text{WHERE } v^2 = \frac{q^2 \langle E^2 \rangle}{m^2 \omega^2 c^2}$$

AFTER THE WAVE PASSES, $v_z \rightarrow 0$ AGAIN.

NONETHELESS, IT IS USEFUL TO NOTE THAT FOR A PLANE WAVE $B_y = E_x$

$$s.o. \quad F_z = \frac{q v_x E_x}{c} = \frac{F_x v_x}{c}$$

BUT $F_x v_x = \frac{dU}{dt}$ = RATE AT WHICH THE FIELD ENERGY IS TRANSFERRED INTO KINETIC ENERGY OF THE CHARGE.

SUPPOSE WE NOW HAVE MANY CHARGES, AND THEY RESIDE IN SOME MATERIAL WHICH CAN DISSIPATE THE ENERGY OF THE MOVING CHARGES (A POOR CONDUCTOR).

THEN THE ENTIRE ENERGY OF THE PLANE WOULD EVENTUALLY BE ABSORBED BY THE MATERIAL.

$$\text{THUS} \quad F_z = \left. \frac{dP_z}{dt} \right|_{\text{IMPARTED TO THE MATERIAL}} = \frac{1}{c} \left. \frac{dU}{dt} \right|_{\text{LOST FROM WAVE}}$$

∴ WE IDENTIFY $P_z = \frac{1}{c} U_{\text{WAVE}}$ IN THE CASE OF TOTAL ABSORPTION.

WHICH IS EXACTLY POYNTING'S RESULT...

IN THE QUANTUM VIEW OF ELECTRO MAGNETIC WAVES, A SINGLE PARTICLE OF LIGHT - THE PHOTON HAS

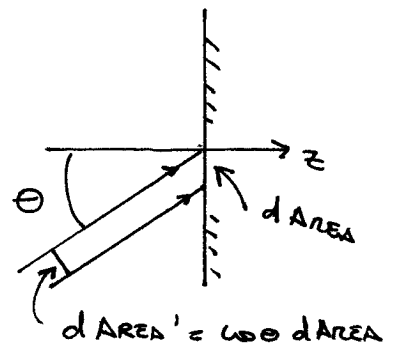
$$\text{ENERGY } U = h\nu \quad \text{AND MOMENTUM } p = \frac{h\nu}{c} = h k$$

AGAIN COMPLETELY CONSISTENT WITH POYNTING'S RELATIONS.

EXAMPLE BLACK BODY RADIATION PRESSURE

SUPPOSE LIGHT IS ISOTROPICALLY INCIDENT ON A PERFECTLY ABSORBING WALL. THIS IS THE SITUATION INSIDE AN IDEALIZED OVEN CONTAINING BLACK BODY RADIATION IN THERMAL EQUILIBRIUM WITH THE WALLS. (A BLACK BODY IS ALSO A PERFECT EMITTER...)

WE WISH TO CALCULATE THE PRESSURE OF THE RADIATION INCIDENT ON THE WALL.



$$P = \frac{\text{FORCE}}{\text{AREA}} = \frac{dP_z}{dt} \frac{1}{\text{AREA}}$$

$$\begin{aligned} dP_z &= \omega \theta \cdot \text{MOMENTUM OF RADIATION WHICH STRIKES THE WALL DURING } dt \\ &= \omega \theta \cdot \underbrace{c \cdot dt \cdot \omega \theta \cdot \text{AREA}}_{d \text{ vol}} \cdot \underbrace{\langle u \rangle / c}_{\text{MOMENTUM DENSITY}} = \langle u \rangle \omega^2 \theta \cdot \text{AREA} \cdot dt \end{aligned}$$

FOR ISOTROPIC INCIDENCE, ALL INTERVALS OF $d\omega \theta$ ARE EQUALLY PROBABLE...

$$\text{so } dP_z \text{ TOTAL} = \langle u \rangle \text{ AREA} \cdot dt \int_0^\pi \underbrace{\cos^2 \theta}_{\text{INTEGRATE OVER A HEMISPHERE}} d\omega \theta = \frac{1}{3} \langle u \rangle \text{ AREA} \cdot dt$$

$$\text{so } \underline{P = \frac{\langle u \rangle}{3}} \quad \left[\text{FOR PERFECTLY REFLECTING, WALLS, THE MOMENTUM TRANSFER DOUBLES} \Rightarrow P = \frac{2}{3} \langle u \rangle \right] \quad \text{AND PERFECTLY EMITTING}$$

THIS MAY BE COMPARED WITH THERMODYNAMICAL RELATIONS

$$P = nKT$$

$n =$ NUMBER OF PARTICLES PER UNIT VOLUME

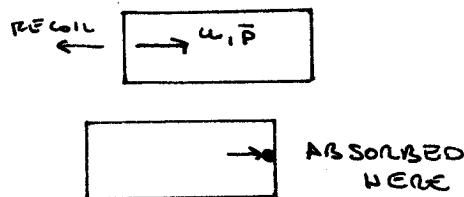
$$\text{AND } \langle u \rangle = n \left(\frac{NKT}{2} \right) \quad \text{WHERE } N = \# \text{ OF DEGREES OF FREEDOM}$$

FOR 'PARTICLES' OF LIGHT, WHICH CAN MOVE IN 3 DIMENSIONS THERE ARE 3 DEGREES OF FREEDOM $\Rightarrow \langle u \rangle = \frac{3}{2} nKT = \frac{3}{2} P$

OR $P = \frac{2}{3} \langle u \rangle$, CORRESPONDING TO OUR RESULT FOR REFLECTING WALLS. [A BLACKBODY RADIATES AS MUCH AS IT ABSORBS, SO THE BACK PRESSURE FROM THE RADIATION IS ALSO $\langle u \rangle/3$]

EXAMPLE $E = mc^2$

WE HAVE A BOX INITIALLY AT REST. SOMETIME THE BOX EMITS LIGHT CARRYING TOTAL ENERGY U FROM THE LEFT WALL SUCH THAT THE LIGHT HEADS FOR THE RIGHT WALL.



WE NOW UNDERSTAND THAT THE LIGHT CARRIES MOMENTUM $p = \frac{U}{c}$.

TO CONSERVE MOMENTUM THE BOX RECOILS TO THE LEFT WITH VELOCITY v SUCH THAT $M_{\text{BOX}} v = \frac{U}{c}$ ($v \ll c$)

AFTER TIME $t = \frac{l}{c}$ THE LIGHT HITS THE RIGHT WALL AND IS ABSORBED,

WHICH ALSO STOPS THE MOTION OF THE BOX. SO THE BOX HAS

MOVED DISTANCE $d = vt = \frac{U l}{M_{\text{BOX}} c^2}$

THIS IS VERY ODD. THE BOX APPEARS TO BE BACK IN ITS ORIGINAL STATE - BUT ITS C.M. HAS MOVED.

EINSTEIN REMARKED THAT THE BOX HAS CHANGED! ENERGY U HAS BEEN TRANSFERRED FROM THE RIGHT TO THE LEFT WALL.

IF WE WRITE $M = \frac{U}{c^2}$ FOR THAT ENERGY, THEN WE SEE THAT THE LAWS OF MECHANICS ARE FULLY RESPECTED.

MAY WE THEN WRITE $E = mc^2$ IN GENERAL...

EXAMPLE ANGULAR MOMENTUM OF A CIRCULARLY POLARIZED WAVE

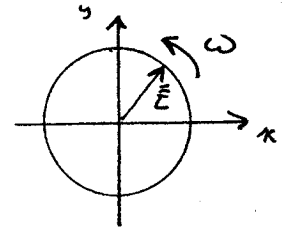
A CIRCULARLY POLARIZED WAVE IS INCIDENT ON AN ABSORBING MATERIAL (AS CONSIDERED IN THE EXAMPLE ON P 128)

THE ELECTRIC FIELD OF SUCH A PLANE WAVE PROPAGATING IN THE Z DIRECTION IS

$$\begin{aligned}\vec{E} &= E \hat{x} \cos(kz - \omega t) \pm E \hat{y} \sin(kz - \omega t) \\ &= \text{Re} \left[E (\hat{x} \mp i \hat{y}) e^{i(kz - \omega t)} \right]\end{aligned}$$

HERE WE CONSIDER THE CASE WITH $\hat{x} + i \hat{y}$ WHICH IS CALLED LEFT HANDED CIRCULAR POLARIZATION

AT $z=0$, $\vec{E} = E \hat{x} \cos \omega t - E \hat{y} \sin \omega t$

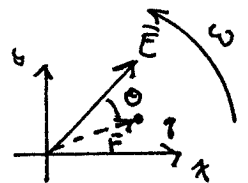


CONSIDER THE EFFECT OF THIS FIELD ON AN ELECTRON IN THE ABSORBER. NEAR $t=0$ THE ELECTRON IS PUSHED IN THE X DIRECTION, BUT SOON THE FIELD DIRECTION CHANGES AND STARTS PUSHING THE CHARGE IN THE Y DIRECTION, ETC.

THE STEADY-STATE RESPONSE OF THE ELECTRON IS THAT IT ROTATES IN A CIRCLE WITH ANGULAR VELOCITY ω , BUT WITH A POSSIBLE LAG IN ANGLE BY SOME AMOUNT Θ , IF THERE IS DISSIPATION.

IF SO, THEN THERE IS A TORQUE ON THE CHARGE

$$\vec{N} = \vec{r} \times q \vec{E} = q r E \sin \Theta$$



HENCE THE FIELD MUST BE POURING ANGULAR MOMENTUM INTO THE ABSORBER!

IT IS ALSO POURING ENERGY IN: $\frac{dU}{dt} = N\omega = \omega \frac{dL}{dt}$

HENCE WE IDENTIFY $L = \frac{U}{\omega}$ AS THE ANGULAR MOMENTUM GIVEN THE ABSORBER AFTER ENERGY U HAS BEEN TRANSFERRED.

FOR TOTAL ABSORPTION, WE CONCLUDE $L_{\text{WAVE}} = \frac{U_{\text{WAVE}}}{\omega}$

NOTE THAT \vec{L} IS IN THE $+\hat{z}$ DIRECTION. (FOR LEFT HANDED POLARIZATION)

THUS, $\vec{L}_{\text{WAVE}} = \frac{U}{\omega} \hat{k} = \frac{\vec{P} c}{\omega} = \frac{\vec{P}}{k}$ WHERE \vec{P} = WAVE MOMENTUM.

WE COMPARE AGAIN WITH THE Q.M. RELATIONS FOR PHOTONS $U = \hbar \omega$
 $\vec{P} = \hbar \vec{k}$
 FOR CIRCULAR POLARIZATION $\rightarrow \vec{L} = \pm \hbar \hat{k}$

UNUSUAL SUPERPOSITIONS OF PLANE WAVES

WE FOUND THAT FOR A SINGLE PLANE WAVE THAT $\vec{E} \perp \vec{B}$. OF POSSIBLE AMUSEMENT ARE WAVES WITH $\vec{E} \parallel \vec{B}$, OBTAINED BY SUITABLE SUPERPOSITION.

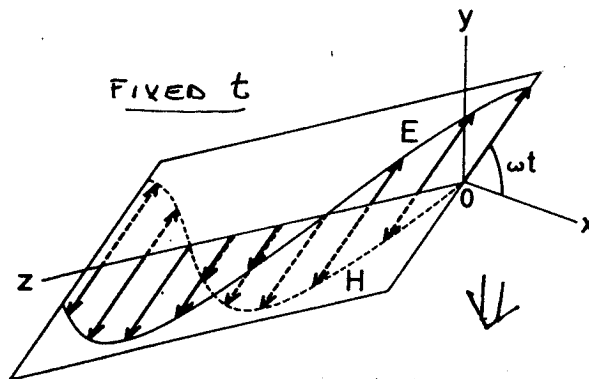
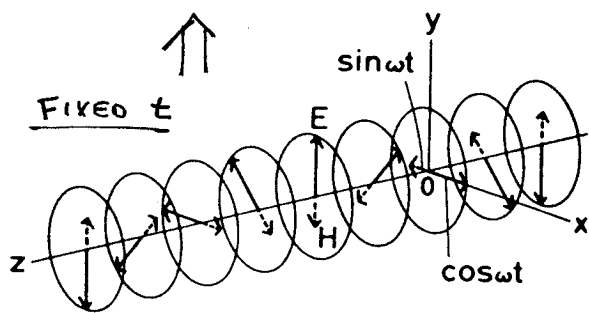
EXAMPLE ADD TWO CIRCULARLY POLARIZED TRAVELLING WAVES MOVING OPPOSITELY:

$$\vec{A} = \frac{A_0}{2} [\sin(kz - \omega t) + \sin(kz + \omega t)] \hat{x} + \frac{A_0}{2} [\cos(kz - \omega t) + \cos(kz + \omega t)] \hat{y} = A_0 \sin kz \cos \omega t \hat{x} + A_0 \cos kz \cos \omega t \hat{y}$$

$$\text{THEN } \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = k A_0 \sin kz \sin \omega t \hat{x} + k A_0 \cos kz \sin \omega t \hat{y}$$

$$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A_y}{\partial z} \hat{x} + \frac{\partial A_x}{\partial z} \hat{y} = k A_0 \sin kz \cos \omega t \hat{x} + k A_0 \cos kz \cos \omega t \hat{y} = \cos \omega t \vec{E}$$

AT A FIXED TIME \vec{E} AND \vec{B} HAVE A FIXED RATIO OF MAGNITUDES EVERYWHERE, AND THEIR DIRECTION VARIES ALONG THE z-AXIS IN A HELICAL FASHION:



A CLOSELY RELATED EXAMPLE IS:

$$A = \frac{A_0}{2} [\sin(kz - \omega t) - \sin(kz + \omega t)] \hat{x} + \frac{A_0}{2} [\cos(kz - \omega t) + \cos(kz + \omega t)] \hat{y} = -A_0 \cos kz \sin \omega t \hat{x} + A_0 \cos kz \cos \omega t \hat{y}$$

$$\vec{E} = k A_0 \cos kz \cos \omega t \hat{x} + k A_0 \cos kz \sin \omega t \hat{y} ; \vec{B} = k A_0 \sin kz \cos \omega t \hat{x} + k A_0 \sin kz \sin \omega t \hat{y} = \tan kz \vec{E}$$

AT FIXED TIME \vec{E} AND \vec{B} POINT IN THE SAME DIRECTION, BUT THEIR MAGNITUDES VARY WITH z

EXAMPLE A SUITABLE SUPERPOSITION OF A SINGLE TRAVELLING WAVE AND A CONSTANT FIELD CAN ALSO OBEY $\vec{E} \parallel \vec{B}$.

$$\left. \begin{aligned} \vec{E} &= E_0 (1 + \cos(kz - \omega t)) \hat{x} - E_0 \sin(kz - \omega t) \hat{y} \\ \vec{B} &= E_0 \sin(kz - \omega t) \hat{x} - E_0 (1 - \cos(kz - \omega t)) \hat{y} \end{aligned} \right\} \text{SATISFIES } \vec{E} \times \vec{B} = 0$$

EVEN MORE AMUSING IS

$$\begin{aligned} \vec{E} &= E_0 (1 + \cos[kz - \omega t]) \hat{x} - E_0 \sin[kz - \omega t] \hat{y} \\ \vec{B} &= E_0 \sin[kz - \omega t] \hat{x} - E_0 (1 - \cos[kz - \omega t]) \hat{y} \end{aligned}$$

FOR $\epsilon \ll 1$ THIS BECOMES

$$\begin{aligned} \vec{E} &\approx E_0 (1 - \epsilon^2 - 2\epsilon^2(kz - \omega t)^2) \hat{x} + E_0 \epsilon^2 (kz - \omega t)^2 \hat{y} \\ \vec{B} &\approx E_0 \epsilon^2 (kz - \omega t)^2 \hat{x} + E_0 (1 - \epsilon^2 - 2\epsilon^2(kz - \omega t)^2) \hat{y} \end{aligned}$$

SEE K. SHIMODA ET AL., AM. J. PHYS. 58, 394 (1990).

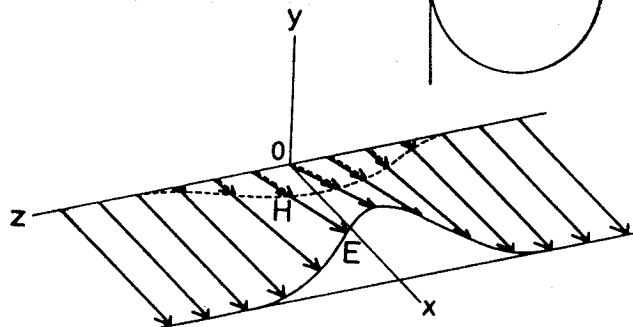
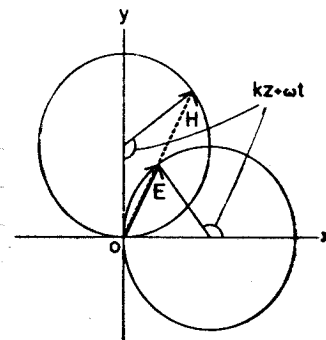


Fig. 4. Superposition of a static electric field and a Gaussian pulse that constitutes parallel electric and magnetic fields as shown by solid and broken arrows, respectively.