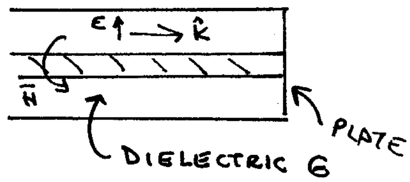
Ph 206 LECTURE 14

WAVES IN BOXES AND PIPES

SUPPOSE WE SEND WAVES DOWN A COAXIAL CABLE WHICH IS TERMINATED BY A METAL PLATE. THE FIELDS $\bar{E}$ AND $\bar{H}$ ARE TRANSVERSE TO $\hat{k}$, THE DIRECTION OF PROPAGATION. SO WHEN THE FIELD HITS THE PLATE IT IS REFLECTED, AS FOR PLANE WAVES INCIDENT ON A CONDUCTING SURFACE.



CLEARLY WE COULD ADD A SECOND PLATE ON THE LEFT, TRAPPING THE FIELD INSIDE A CAVITY. THE FIELD BOUNCES BACK AND FORTH (AND EVENTUALLY DIES OUT DUE TO JOULE LOSSES IN THE CONDUCTORS). THE BOUNCING OF TRAVELLING WAVES IS EQUIVALENT TO <u>STANDING WAVES</u>

i.e., $\bar{E} = \bar{E}(x,y,z)\, e^{-i\omega t}$ FOR OSCILLATORY TIME DEPENDENCE.

FURTHERMORE THE BOUNDARY CONDITIONS ON WAVES AT CONDUCTING SURFACES ALLOW ONLY CERTAIN FREQUENCIES (AND WAVELENGTHS) TO SURVIVE. FOR EXAMPLE, $\bar{E} = 0$ INSIDE A PERFECT CONDUCTOR, SO $\bar{E}_{\parallel} = 0$ AT THE BOUNDARY. BUT THE $\bar{E}$ FIELD OF THE WAVE IS ALWAYS $\parallel$ TO THE END PLATE. THUS IF THE CAVITY EXTENDS FROM $z = 0$ TO $z = l$,

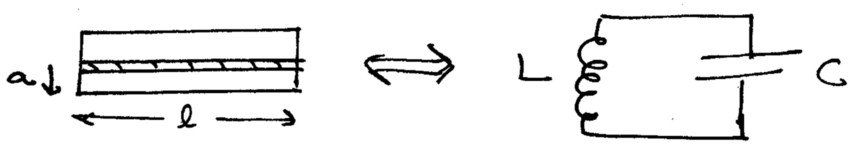
$$\bar{E} \sim \bar{E}(x,y) \sin \frac{n\pi z}{l}\, e^{-i\omega t}$$

OR $K = \frac{n\pi}{l} = \sqrt{\epsilon}\,\frac{\omega}{c}$ $\Rightarrow$ $\omega = \frac{1}{\sqrt{\epsilon}} \frac{n\pi c}{l}$ $\quad n = 1, 2, 3 \ldots$

↖ VIA WAVE EQUATION

CIRCUIT ANALOGY

THE CAVITY FORMED BY ADDING METAL PLATES AT THE ENDS OF A LENGTH l OF COAXIAL CABLE HAS ELECTRICAL PROPERTIES SIMILAR TO AN EQUIVALENT CIRCUIT



THERE IS A 1-TURN SELF INDUCTANCE $L = \frac{2l}{c^2} \ln b/a$

WHICH IS IN SERIES WITH THE CAPACITANCE BETWEEN THE INNER AND OUTER CONDUCTORS

$$C = \frac{l\epsilon}{2 \ln b/a}$$

THE CIRCUIT EQUATION $L\ddot{I} + \frac{I}{C} = 0 \Rightarrow$ NATURAL FREQUENCY $\omega = \frac{1}{\sqrt{LC}} = \frac{c}{\ell\sqrt{\epsilon}}$

WHICH IS QUALITATIVELY SIMILAR TO OUR RESULT $\omega = \frac{\pi c}{\sqrt{\epsilon}\,\ell}$ FOR THE LOWEST FREQUENCY FOUND ON p 158.

THE CIRCUIT ANALOGY IS USEFUL FOR GETTING AN APPROXIMATE UNDERSTANDING OF THE SIMPLEST MODE OF A CAVITY.

MORE ON THE BOUNDARY CONDITIONS

IN A REAL CONDUCTOR THE FIELDS PENETRATE DISTANCE $d \sim$ SKIN DEPTH INTO THE CONDUCTOR. ONLY FOR PERFECT CONDUCTORS ARE THE FIELDS COMPLETELY EXCLUDED. FURTHER, WE SAW THAT THE COMPONENTS WHICH PENETRATE ARE MAINLY $E_{\parallel}$ AND $H_{\parallel}$ AND $|E_{\parallel}| = \frac{\omega d}{\sqrt{2}\,c}|H_{\parallel}| \ll |H_{\parallel}|$

$E_{\perp}$ VANISHES INSIDE THE CONDUCTOR DUE TO CHARGES WHICH COLLECT ON THE SURFACE TO TERMINATE THE FIELD LINES. BUT $E_{\perp}$ CAN BE BIG OUTSIDE THE CONDUCTOR.

SINCE $\bar{\nabla}\cdot\bar{B} = 0$, $B_{\perp}$ IS CONTINUOUS ACROSS THE SURFACE. IN THE LIMIT THAT THE FIELDS ARE EXCLUDED FROM THE CONDUCTOR $B_{\perp}$ MUST VANISH OUTSIDE ALSO.

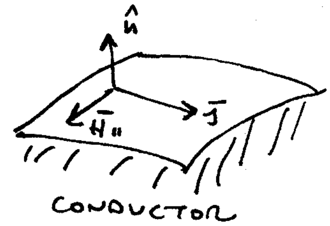
$\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t} \Rightarrow E_{\parallel}$ CONTINUOUS SO LONG AS THERE IS NO DELTA FUNCTION OF $\bar{B}$ ALONG THE SURFACE — WHICH CLEARLY CAN'T HAPPEN.

$\oint E\cdot d\ell = -\frac{1}{c}\frac{d}{dt}\Phi_M \sim 0$

SO IN THE LIMIT OF A PERFECT CONDUCTOR $E_{\parallel}$ VANISHES OUTSIDE ALSO.

FINALLY $\bar{\nabla}\times\bar{H} = \frac{4\pi}{c}\bar{J} + \frac{1}{c}\frac{\partial\bar{D}}{\partial t}$

For a real conductor in which the currents are spread over a layer of thickness d, we can still say $H_{\parallel}$ is continous at the surface. But as $d \to 0$ this does NOT mean $H_{\parallel} \to 0$ outside. Rather the currents become squeezed into an infinitesimal layer at the surface $\Rightarrow$ "surface currents" $H_{\parallel}$ can remain finite on the dielectric side of the boundary.

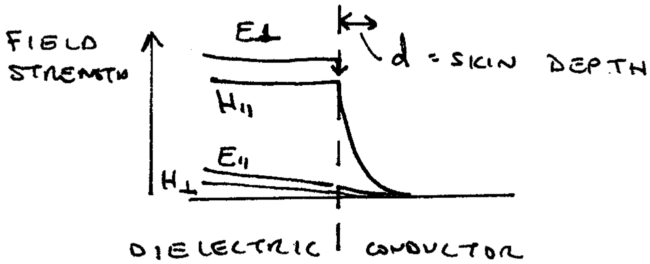


Ampere's law then tells us

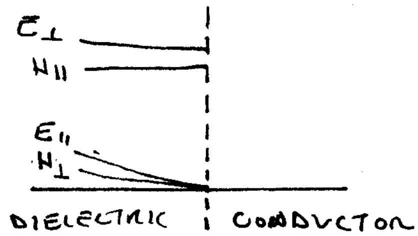
$$\bar{H}_{\parallel}\Big|_{\text{outside}} = \frac{4\pi}{c}\,\bar{J}\times\hat{n}$$

or $$\bar{J} = \frac{c}{4\pi}\,\hat{n}\times\bar{H}_{\parallel}$$

where $\hat{n}$ = outward normal



REAL CONDUCTOR



IDEAL CONDUCTOR

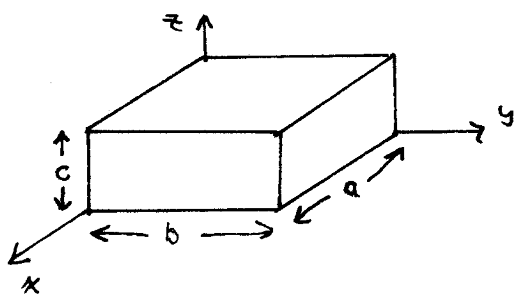
Rectangular Cavity

One of the most familiar boxes of electromagnetic waves is the microwave oven – a rectangular cavity.

Suppose we solve for the possible $\bar{E}$ and $\bar{H}$ directly from Maxwell's equations.

Inside the box we have the free space wave equations

$$\nabla^2\begin{Bmatrix}\bar{E}\\ \bar{H}\end{Bmatrix} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\begin{Bmatrix}\bar{E}\\ \bar{H}\end{Bmatrix}$$



We want standing wave solutions

$$\bar{E} = \bar{E}(x,y,z)\,e^{-i\omega t} \Rightarrow \nabla^2\bar{E} = \frac{-\omega^2}{c^2}\bar{E} \quad \text{etc.}$$

FOR EACH COMPONENT OF $\bar{E}$ OR $\bar{H}$ WE HAVE AN EQUATION LIKE

$$\nabla^2 \phi = -\frac{\omega^2}{c^2}\phi$$

ALMOST LAPLACE'S EQUATION.

WHY NOT TRY A SEPARATION OF VARIABLE SOLUTION

$$\phi = X(x)\,Y(y)\,Z(z)$$

$$\Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\frac{\omega^2}{c^2}$$

THIS IS EVEN NICER THAN LAPLACE'S EQUATION AS NOW WE CAN SET ALL 3 SEPARATION CONSTANTS NEGATIVE AND GET ONLY OSCILLATORY FUNCTIONS

$$\frac{X''}{X} = -k_1^2 \qquad \frac{Y''}{Y} = -k_2^2 \qquad \frac{Z''}{Z} = -k_3^2 \qquad \text{WHERE} \quad k_1^2 + k_2^2 + k_3^2 = \frac{\omega^2}{c^2}$$

THUS
$$E_i = E_{0i} \begin{Bmatrix} \sin k_1 x \\ \cos k_1 x \end{Bmatrix} \begin{Bmatrix} \sin k_2 \\ \cos k_2 \end{Bmatrix} \begin{Bmatrix} \sin k_3 \\ \cos k_3 \end{Bmatrix} e^{-i\omega t}$$

THE BOUNDARY CONDITIONS ON $\bar{E}$ TELL US

$$\bar{E}_{\parallel} = 0 \text{ AT BOUNDARY} \Rightarrow \begin{cases} E_x = 0 & \text{AT } y = 0, b \ \& \ z = 0, c \\ E_y = 0 & \text{AT } x = 0, a \ \& \ z = 0, c \\ E_z = 0 & \text{AT } x = 0, a \ \& \ y = 0, b \end{cases}$$

SO
$$E_x = E_{0x} \begin{Bmatrix} \sin k_1 x \\ \cos k_1 x \end{Bmatrix} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}\, e^{-i\omega t}$$

$$E_y = E_{0y} \sin\frac{l\pi x}{a} \begin{Bmatrix} \sin k_2 y \\ \cos k_2 y \end{Bmatrix} \sin\frac{n\pi z}{c}\, e^{-i\omega t}$$

$$E_z = E_{0z} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \begin{Bmatrix} \sin k_3 z \\ \cos k_3 z \end{Bmatrix} e^{-i\omega t}$$

LIKEWISE $\bar{H}_\perp = 0$ AT BOUNDARY SO $H_x = 0$ AT $x = 0, a$ ETC

$$H_x = H_{0x} \sin\frac{l\pi x}{a} \begin{Bmatrix} \sin k_2 y \\ \cos k_2 y \end{Bmatrix} \begin{Bmatrix} \sin k_3 z \\ \cos k_3 z \end{Bmatrix} e^{-i\omega t}$$

ETC.

BUT ALSO
$$\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial H}{\partial t} = \frac{i\omega}{c}\bar{H}$$

$$\Rightarrow \frac{i\omega}{c}\bar{H}_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \quad \text{ETC.}$$

$$\frac{i\omega}{c} H_{0x} \sin\frac{l\pi x}{a} \begin{Bmatrix} \sin k_2 y \\ \cos k_2 y \end{Bmatrix} \begin{Bmatrix} \sin k_3 z \\ \cos k_3 z \end{Bmatrix} = \frac{m\pi}{b} E_{0y} \sin\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \begin{Bmatrix} \sin k_3 z \\ \cos k_3 z \end{Bmatrix}$$

$$- \frac{n\pi}{c} E_{0z} \sin\frac{l\pi x}{a} \begin{Bmatrix} \sin k_2 y \\ \cos k_2 y \end{Bmatrix} \cos\frac{n\pi z}{c}$$

CLEARLY OUR CHOICES ARE RESTRICTED TO

$$E_x = E_{0x} \cos\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} e^{-i\omega t}$$

$$E_y = E_{0y} \sin\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \sin\frac{n\pi z}{c} e^{-i\omega t}$$

$$E_z = E_{0z} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \cos\frac{n\pi z}{c} e^{-i\omega t}$$

$$H_x = H_{0x} \sin\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \cos\frac{n\pi z}{c} e^{-i\omega t}$$

$$H_y = H_{0y} \cos\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \cos\frac{n\pi z}{c} e^{-i\omega t}$$

$$H_z = H_{0z} \cos\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \sin\frac{n\pi z}{c} e^{-i\omega t}$$

WRITING $\bar{K} = \left(\frac{l\pi}{a}, \frac{m\pi}{b}, \frac{n\pi}{c}\right)$ $\qquad K^2 = \frac{\omega^2}{c^2}$

AND $i\frac{\omega}{c}\bar{H} = \nabla \times \bar{E} \Rightarrow \bar{H}_0 = \hat{K} \times \bar{E}_0$ RELATES THE COEFFICIENTS

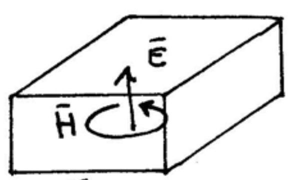
THUS $\bar{E}$ AND $\bar{H}$ ARE $\perp$ EVERYWHERE.

FINALLY $\bar{\nabla}\cdot\bar{E} = 0 = \bar{\nabla}\cdot\bar{H} \Rightarrow \bar{K}\cdot\bar{E}_0 = 0 = \bar{K}\cdot\bar{H}_0$

SO $\bar{E}$, $\bar{H}$ AND $\bar{K}$ ARE MUTUALLY ORTHOGONAL, AS FOR A PLANE WAVE. BUT $\bar{E}$ AND $\bar{H}$ ARE COMPLICATED — AND DO <u>NOT</u> RESEMBLE STATIC SOLUTIONS.

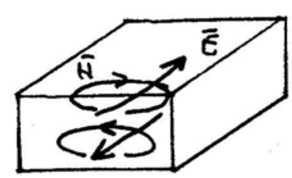
THE INDICES (l, m, n) LABELLING THE FIELD CONFIGURATIONS MUST BE POSITIVE INTEGERS (OR ONE OF THEM MAY BE ZERO.) HOWEVER FOR EACH SET (l, m, n) TWO DISTINCT FIELD CONFIGURATIONS ARE POSSIBLE: $\bar{E}\cdot\bar{K} = 0$ CAN BE SATISFIED BY TWO ORTHOGONAL VECTORS IN GENERAL. (ONLY 1 IF $l, m,$ OR $n = 0$). THESE FIELD CONFIGURATIONS ARE OFTEN CALLED THE <u>NORMAL</u> <u>MODES</u> OF THE CAVITY. FOR EACH (l, m, n) THERE ARE 2 MODES - OF ORTHOGONAL POLARIZATION.

$(1, 1, 0)$ $\quad E_x = E_y = 0$, $H_z = 0$

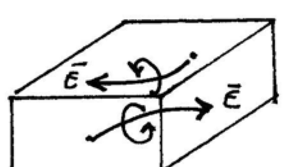


OK FOR COOKING ↗

$(0, 1, 2) \rightarrow$

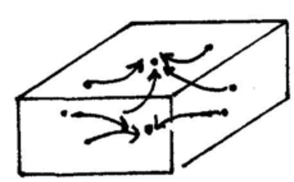


$(1, 1, 1)_a \rightarrow$



WITH $\bar{E}_0 = (1, -1, 0)$

$(1, 1, 1)_b \rightarrow$



WITH $\bar{E}_0 = (1, 1, -2)$ $\qquad \bar{E}$ ONLY SHOWN

$\bar{H}$ ALWAYS LOOPS AROUND $\bar{E}$. AT THE WALLS: $\bar{E}$ IS $\perp$.
$\bar{H}$ IS $\parallel$.

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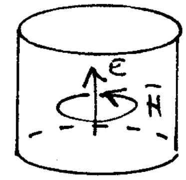
OTHER CAVITIES



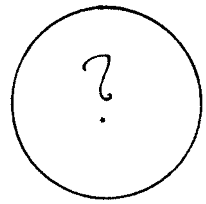
A SIMPLE MODE OF A CYLINDRICAL CAN IS SHOWN.

ON THE HOMEWORK SET WE ENCOURAGE YOU TO SOLVE THIS BY ITERATION:

1. START WITH $\bar{E} = \bar{E}_0 e^{-i\omega t}$ UNIFORM IN SPACE
2. THIS INDUCES $\bar{B}_1$ WHICH CIRCULATES
3. $\bar{B}_1$ INDUCES $\bar{E}_1$ WHICH TENDS TO OPPOSE $\bar{E}_0$
4. $\bar{E}_1$ INDUCES $\bar{B}_2$ ....

THE SERIES CONVERGES .... TO A BESSEL FUNCTION!

THE SOLUTIONS INSIDE A HOLLOW SPHERE CAN BE QUITE COMPLICATED. AFTER LECTURE 16 WE CAN CONSTRUCT AN INTERESTING CLASS OF SOLUTIONS BY A SUPERPOSITION TRICK.

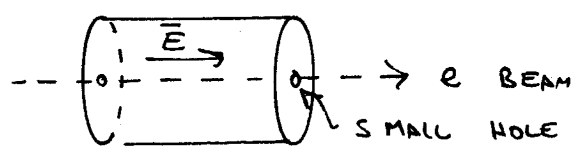


WE ADD AN IMAGINARY ELECTRIC DIPOLE AT THE CENTER. AS THIS OSCILLATES IT RADIATES FIELDS OUTWARDS.

MATHEMATICALLY THERE ALSO EXIST WAVES WHICH CONVERGE ONTO A DIPOLE AT THE CENTER. SO WE ADD SUCH A SOLUTION WHICH CONVERGES ONTO A DIPOLE OPPOSITE TO THE FIRST ONE. BY SUPERPOSITION WE ARE LEFT WITH NO DIPOLE BUT LOTS OF WAVES .....

OTHER THAN FOR COOKING, CAVITIES ARE MOSTLY USED FOR TRANSFERRING ENERGY BETWEEN ELECTRONS AND WAVES

IF THE DENSITY OF ELECTRONS IN THE BEAM OSCILLATES WITH A RESONANT FREQUENCY OF THE CAVITY, THEN ALL THE ELECTRONS CAN BE ACCELERATED AS THEY PASS THRU THE CAVITY — JUST AS IF THE FIELD WERE D.C.



OR, BY CHANGING THE RELATIVE PHASE OF THE CAVITY FIELD AND THE BEAM, THE BEAM WOULD BE DECELERATED.

EITHER WAY THERE IS A TRANSFERENCE OF ENERGY.

IN <u>KLYSTRONS</u> THE BEAM ENERGY IS GIVEN TO THE FIELD — USING A LARGE NUMBER OF LOW ENERGY ELECTRONS.

IN <u>PARTICLE ACCELERATORS</u> THE FIELD ENERGY IS TRANSFERRED TO OTHER CAVITIES WHICH THEN GIVE THE ENERGY BACK TO A SMALL NUMBER OF HIGH ENERGY PARTICLES

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WAVES IN PIPES

VERY INTENSE FIELDS CAN BE CREATED INSIDE CAVITIES BY THE ELECTRON PUMPING SKETCHED ABOVE. OFTEN IT IS DESIREABLE TO MOVE THAT ENERGY SOMEWHERE ELSE. THE TWO WIRE CABLE IS NOT GOOD BECAUSE THE FIELDS WOULD BE EXPOSED TO THE OUTSIDE WORLD ⇒ SPARKING, MELTING ETC. FOR VERY HIGH FIELDS.

THE COAXIAL CABLE SUFFERS FROM A MECHANICAL PROBLEM - HOW DO YOU SUPPORT THE INNER CONDUCTOR. FOR VERY HIGH FIELDS THE SUPPORT MECHANISM LEADS TO LOSSES, ETC.

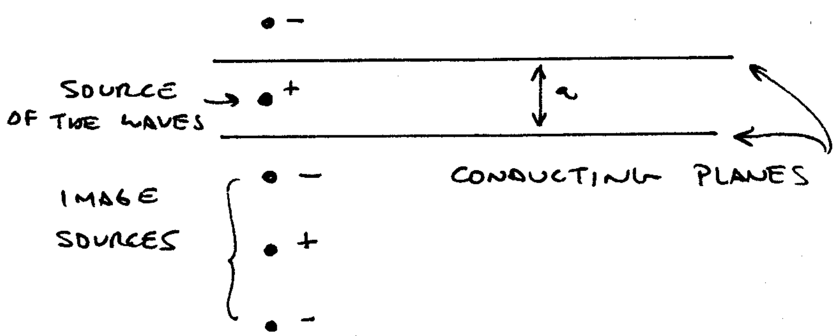
THE BEST SOLUTION IS TO TRANSPORT THE FIELDS INSIDE HOLLOW PIPES. AS MENTIONED IN LECTURE 13, WE CANNOT SEND PURELY TRANSVERSE WAVES INTO HOLLOW PIPES — AS THE TRANSVERSE FIELDS MUST SATISFY ELECTRO - AND MAGNETO STATICS.

(A PUZZLE - IF YOU LOOK THRU A WATER PIPE YOU CAN STILL SEE THE LIGHT ???)

BUT IT TURNS OUT WE CAN SEND THE WAVE THRU THE PIPE IF WE SEND THEM IN CROOKED - SO THEY FOLLOW A ZIG-ZAG PATH BOUNCING OF THE WALLS.

WE CAN GET A SENSE OF HOW IT WORKS BY CONSIDERING A 'PIPE' WHICH IS JUST TWO PARALLEL CONDUCTING PLANES.

WE SUPPOSE THE WAVES EMANATE FROM A 'SOURCE' LOCATED BETWEEN THE PLATES. FOR EXAMPLE, THE SOURCE MIGHT BE A LONG WIRE IN WHICH WE FORCE THE CHARGES TO OSCILLATE AT FREQUENCY ω.



BY AN EXTENSION OF THE IMAGE METHOD, THE FIELDS DOWN THE PIPE CAN BE THOUGHT OF AS DUE TO A WHOLE SERIES OF IMAGE SOURCES, IGNORING THE CONDUCTORS.

THE POLARITY OF THE SOURCES WILL ALTERNATE AS SHOWN.

WE CAN NOW DO A KIND OF HUYGEN'S CONSTRUCTION FOR WAVEFRONTS EMANATING FROM THE SOURCES

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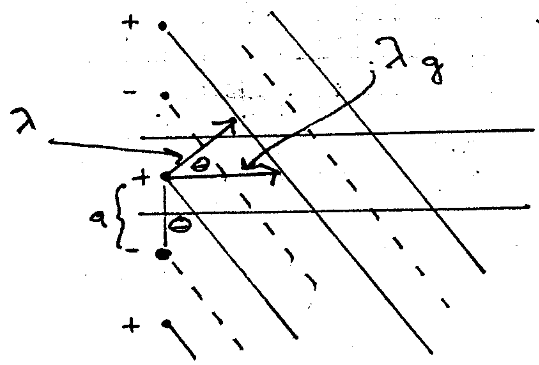


CONSIDER WAVE FRONTS AT ANGLE θ AS SHOWN.

THE VARIOUS WAVES INTERFERE CONSTRUCTIVELY ONLY IF

$$\sin\theta = \frac{\lambda}{2a}$$

IF WE TRY $\theta = 0$ – I.E. SENDING THE WAVES STRAIGHT IN, WE GET DESTRUCTIVE INTERFERENCE (AT LARGE DISTANCES FROM THE SOURCE). THE HEAD ON WAVES DIE OUT RAPIDLY AND DO NOT PROPAGATE.

BUT IF $\lambda < 2a$ CONSTRUCTIVE INTERENCE IS POSSIBLE.
I.E., SHORT WAVES CAN PROPAGATE!

WE DEFINE λ_g = <u>GUIDE WAVELENGTH</u> = DISTANCE BETWEEN CRESTS AS MEASURED ALONG THE GUIDE

THEN $$\cos\theta = \frac{\lambda}{\lambda_g} \Rightarrow \lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} > \lambda$$

NOTE THAT IF THE WAVE FRONTS MOVE AT VELOCITY c, THE INTERCEPT OF THE WAVE FRONTS WITH THE CONDUCTOR MOVE AT VELOCITY $$\frac{c}{\cos\theta} = \frac{c}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} > c \;!!$$

WE CALL THIS THE <u>PHASE VELOCITY</u> OF THE WAVE. IT IS GREATER THAN c DUE TO THE ZIG-ZAG GEOMETRY, AND DOES <u>NOT</u> REPRESENT THE VELOCITY WITH WHICH SIGNALS COULD BE SENT.

WE CAN RELATE THIS TO OUR USUAL DEFINITION OF PHASE VELOCITY AS FOLLOWS:

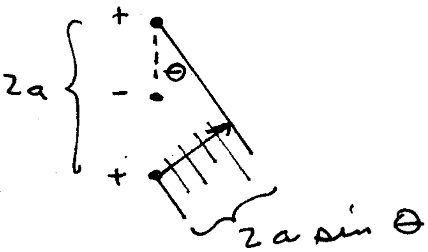
DEFINE $$K_g = \frac{2\pi}{\lambda_g} = \frac{2\pi}{\lambda}\sqrt{1-\left(\frac{\lambda}{2a}\right)^2} = K\sqrt{1-\left(\frac{\lambda}{2a}\right)^2} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{K\lambda}{2a}\right)^2}$$
$$= \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2}$$ USING $K = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

THUS $$\omega = c\sqrt{K_g^2 + \left(\frac{\pi}{a}\right)^2}, \text{ so } V_p = \frac{\omega}{K_g} = c\sqrt{1+\left(\frac{\pi}{aK_g}\right)^2} > c,$$

BUT $$\underset{\substack{\uparrow \\ \text{GROUP}}}{V_g} = \frac{d\omega}{dK_g} = c\Big/\sqrt{1+\left(\frac{\pi}{aK_g}\right)^2} < c, \text{ AND } V_p V_g = c^2.$$

We remark that the condition for constructive interference may be generalized. It is sufficient that the distance $2a \sin\Theta$ be an integral number of wavelengths

i.e. $\sin\Theta = \dfrac{n\lambda}{2a}$



which leads at once to $k_g = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{a}\right)^2}$

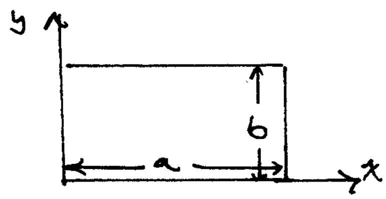
Thus for each frequency we desire to send into the pipe, there is a sequence of possible ways to do this – corresponding to bouncing off the walls at various angles. These are called the <u>wave guide</u> <u>modes</u>

Another feature can be noted from our geometrical analysis. $\bar{E}$ and $\bar{H}$ are both perpendicular to the direction of motion of the zig-zag waves. But as viewed along the guide axis, at most one of $\bar{E}$ or $\bar{H}$ can be transverse. Either can be arranged, which leads to the description:

TE = transverse electric waves

TM = transvers magnetic waves

<u>Solution for the wave guide fields</u>



We consider a guide of rectangular cross section – by far the most important in practice.

<u>TE waves</u>. We look for waves of the type

$$\bar{E} = \bar{E}_\perp(x,y)\, e^{i(k_g z - \omega t)} \qquad (\text{i.e., } E_z = 0)$$

The boundary condition is that $\bar{E}_\parallel$ must vanish at the walls.

From our analysis of the rectangular cavity we quickly anticipate

$$E_x = E_{0x} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$E_y = E_{0y} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

which is O.K. at the boundaries.

WHAT ABOUT THE WAVE EQUATION: $\nabla^2 \bar{E} = \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}$

PLUG IN: $-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 - k_g^2 = -\frac{\omega^2}{c^2}$

OR $k_g = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$ AS EXPECTED.

WE CAN FIND $\bar{H}$ VIA $\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t} = \frac{i\omega}{c}\bar{H}$

SO $H_x = \frac{-c}{i\omega}\frac{\partial E_y}{\partial z} = -\frac{ck_g}{\omega}E_y = -\frac{ck_g}{\omega}E_{oy}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$

$H_y = \frac{c}{i\omega}\frac{\partial E_x}{\partial z} = \frac{ck_g}{\omega}E_{ox}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

$H_z = \frac{c}{i\omega}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\frac{ic}{\omega}\left(\frac{m\pi}{a}E_{oy} - \frac{n\pi}{b}E_{ox}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$

NOTE THAT $\bar{E}\cdot\bar{H} = 0$ AND $\bar{E} = -\frac{\omega}{ck_g}\hat{z}\times\bar{H}$ [$\bar{\nabla}\cdot\bar{E} = 0 = \bar{\nabla}\cdot\bar{H}$ ADD FURTHER RELATIONS. ONLY ONE CONSTANT, SAY H_{oz}, DETERMINES ALL]

TM WAVES

WE START WITH $\bar{H} = \bar{H}(x,y)\,e^{i(k_g z - \omega t)}$ WITH $H_z = 0$

AND REQUIRE $\bar{H}_\perp$ TO VANISH AT THE WALLS.

THEN $H_x = H_{ox}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$

$H_y = H_{oy}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

AND $\bar{\nabla}\times\bar{H} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t}$ LEADS TO

$E_x = E_{ox}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

$E_y = E_{oy}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$

$E_z = E_{oz}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

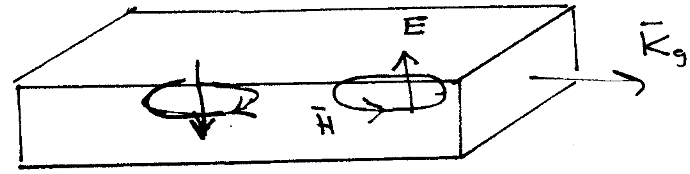
$\Bigg\}\ \bar{H} = \frac{\omega}{ck_g}\hat{z}\times\bar{E}$

NOTE THAT IF m OR $n = 0$, THEN $E_z = 0$ AND THE WAVE BECOMES TEM $\Rightarrow$ IT WON'T PROPAGATE!

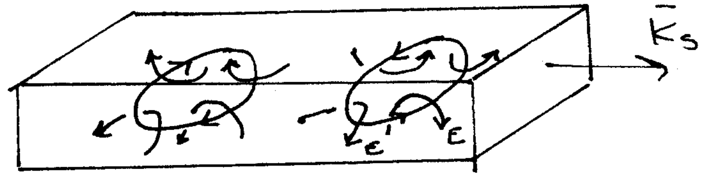
NOW $\omega = c\sqrt{k_g^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

SO THE LOWEST FREQUENCY MODE IS THE TE 1,0 OR TE 0,1

TE 10



TM 11



SOME FINAL REMARKS ON ω, K_g ETC.

$$K_g = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

IF K_g IS IMAGINARY, THE WAVES DIE OUT

SO THERE IS A LOWEST FREQUENCY WHICH CAN PROPAGATE FOR ANY GIVEN M AND n. — AND AN ABSOLUTE LOWEST FREQUENCY

$$\omega_{MIN} = \text{MIN OF } \left(\frac{\pi c}{a}, \frac{\pi c}{b}\right)$$

CLEARLY WAVEGUIDES ARE HIGH FREQUENCY DEVICES!

WRITING $\omega = c\sqrt{K_g^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

WE HAVE $V_{PHASE} = \dfrac{\omega}{K_g} > c$ $\quad V_{GROUP} = \dfrac{d\omega}{dK_g} = \dfrac{c^2 K_g}{\omega}$

AND $V_p V_g = c^2$

POWER LOSS INTO THE WALLS

FOR REAL CONDUCTORS THERE IS SIGNIFICANT POWER LOSS INTO THE WALLS DUE TO JOULE HEATING.

FOR $\omega \sim \omega_{MIN}$ THE LOSSES MUST BE SEVERE, SINCE FOR $\omega < \omega_{MIN}$ ALL THE POWER DISAPPEARS INTO THE WALLS RATHER QUICKLY — THE WAVE DOES NOT PROPAGATE.

BUT FOR $\omega \gg \omega_{MIN}$ THE LOSSES ARE BAD ALSO. THIS IS BECAUSE THE SKIN DEPTH $\to 0$ $\Rightarrow$ VERY HIGH CURRENT DENSITIES $\Rightarrow$ HIGH RAYLEIGH RESISTANCE. $\Rightarrow$ HIGH LOSS.

SO THE GREATEST RATIO OF TRANSMITTED POWER IS OBTAINED AT SOME INTERMEDIATE ω.

FOR TM MODES, THE LOSSES ARE LEAST AT $\omega = \sqrt{3}\, \omega_{MIN}$

FOR TE MODES THE BEST CHOICE OF ω DEPENDS ON THE MODE. HOWEVER THE LOWEST LOSSES OF ALL ARE OBTAINED IN THE TE_{01} MODE WITH $\omega \sim 2.5\, \omega_{MIN}$.

HENCE IN PRACTICE THIS WILL BE THE ONLY MODE USED.

THE POWER CONSIDERATIONS CAN BE MADE QUANTITATIVE USING THE POYNTING VECTOR.

$$\text{POWER LOSS TO WALL} = \langle \bar{S} \rangle_{\perp \text{ TO WALL}} = \frac{c}{8\pi} Re\left(\bar{E} \times \bar{H}^*\right)_\perp$$

$$= \frac{c}{8\pi} Re\left(\bar{E}_\parallel \times \bar{H}^*_\parallel\right) \text{ AT WALL}$$

FROM OUR STUDY OF WAVES IN CONDUCTORS, WE FOUND (P 147 OR 151)

$$\bar{E}_\parallel = -\frac{\omega d}{2c}(1-i)\, \hat{n} \times \bar{H}_\parallel \qquad (\hat{n} \text{ POINTS INTO WALL})$$

$$\text{SO} \quad \langle \bar{S} \rangle_{LOST} = \frac{\omega d}{16\pi} H_\parallel^2\, \hat{n} = \left(\frac{c}{4\pi}\right)^2 \frac{H_\parallel^2}{2\sigma d}\, \hat{n} \qquad \left[d = \frac{c}{\sqrt{2\pi\sigma\omega}}\right]$$

WHICH CAN BE INTEGRATED OVER THE SURFACE TO GET THE TOTAL LOSS.

IN CAVITIES THIS LOSS IS COMPARED TO THE STORED ENERGY

$$U = \frac{1}{8\pi}\int \left(E^2 + H^2\right) d\text{vol}.$$

IN GUIDES, WE WISH TO CALCULATE THE POWER FLOW DOWN THE GUIDE

$$\langle S_z \rangle = \frac{c}{8\pi} Re\left(\bar{E} \times \bar{H}^*\right)_z$$

FOR TE WAVES $\quad \bar{E} = -\frac{\omega}{c k_g}\hat{z} \times \bar{H} = -\frac{k}{k_g}\hat{z} \times \bar{H} \qquad \left(k = \frac{\omega}{c}\right)$

SO $\quad \langle S_z \rangle_{TE} = \left(\frac{c}{4\pi}\right)\frac{k}{k_g}\frac{H_t^2}{2}$ WHERE $\bar{H}_t = \bar{H} - \bar{H}_z$ = TRANSVERSE PART OF $\bar{H}$

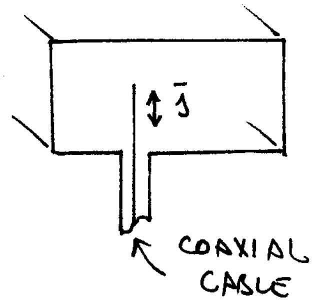
FOR TM WAVES $\quad \bar{H} = \frac{k}{k_g}\hat{z} \times \bar{E}$ WHICH LEADS TO

$$\langle S_z \rangle_{TM} = \left(\frac{c}{4\pi}\right)\frac{k_g}{k}\frac{H_t^2}{2}$$

INTEGRATE THIS OVER THE GUIDE CROSS SECTION....

EXCITING THE WAVEGUIDE MODES

HOW DO WE GET THE WAVES INTO THE GUIDE?



A POSSIBLE DEVICE IS A COAXIAL CABLE CONNECTED TO ONE FACE OF THE GUIDE SO THAT THE CENTER CONDUCTOR OF THE CABLE PROTRUDES INTO THE GUIDE. THEN AN OSCILLATORY CURRENT FED INTO THE CABLE WILL RESULT IN RADIATION INSIDE THE GUIDE, POURING ENERGY INTO THE VARIOUS GUIDE MODES.

WE CAN MAKE A QUANTITAVE ARGUMENT BY A DEEP AND DEVIOUS TRICK: TIME REVERSAL INVARIANCE!

WE WANT TO KNOW THE STRENGTH OF THE FIELD $\bar{E}$ (AND $\bar{H}$) CAUSED BY A GIVEN CURRENT DISTRIBUTION $\bar{J}$.

CONSIDER THE TIME REVERSED SITUATION. A WAVE OF STRENGTH $\bar{E}$ (AND $-\bar{H}$) PROGATES TOWARDS THE WIRE, WHICH NOW CARRIES CURRENT $-\bar{J}$ (SINCE TIME REVERSAL CHANGES THE SIGN OF VELOCITIES BUT NOT CHARGES).

THE INCOMING WAVE CARRIES POWER $\langle P_0 \rangle = \int_{AREA} \langle \bar{S} \rangle = \frac{c}{8\pi} \int \frac{E_t^2}{Z} \, dAREA$

WHERE $E_t = Z H_t$ RELATES THE TRANSVERSE FIELD STRENGTHS

AND $Z = \begin{cases} k/k_g & \text{FOR TE MODES} \\ k_g/k & \text{FOR TM MODES} \end{cases}$ ACCORDING TO p 169

THE POWER ABSORBED BY THE WIRE IS $\langle P_{ABS} \rangle = \frac{1}{2} \int (-\bar{J}) \cdot \bar{E}_t \, dvol$

$\nwarrow$ TIME AVERAGE

(SUPPOSING $\bar{J}$ IS CONFINED TO A TRANSVERSE PLANE)

(RECALL OUR DERIVATION OF THE POYNTING VECTOR IN LECTURE 10.)

HENCE THE TRANSMITTED POWER IS $\langle P_{TRANS} \rangle = \langle P_0 \rangle - \langle P_{ABS} \rangle$

BY TIME REVERSAL INVARIANCE, IF POWER $\langle P_{TRANS} \rangle$ FLOWED TOWARDS THE WIRE (NOW CARRYING CURRENT $\bar{J}$ AGAIN), THE RADIATION DUE TO $\bar{J}$ WOULD ADD TO THE INCIDENT WAVE TO PRODUCE A FINAL WAVE OF POWER $\langle P_0 \rangle$.

THIS IS ALMOST WHAT WE WANT. IF WE COULD JUST GET RID OF THE EXTRA WAVE CARRYING $\langle P_{TRANS} \rangle$, WE'D HAVE IT.

TO DO THIS IT IS USEFUL TO INTRODUCE SOME ADDITIONAL NOTATION

FOR A GIVEN GUIDE MODE DEFINE THE FIELD STRENGTH $\bar{E}_0$ SUCH THAT $\int E_{0t}^2 \, dAREA = 1$

THEN $\bar{E} = a\bar{E}_0$ IN GENERAL. [a = DIMENSIONLESS CONSTANT ; NOT AREA]

WITH THIS NOTATION $\langle P_0 \rangle = \frac{c}{8\pi} \frac{a^2}{Z}$

$$\langle P_{ABS} \rangle = -\frac{a}{2} \int \bar{J} \cdot \bar{E}_{0t}$$

SUPPOSE FURTHER THAT $\langle P_{ABS} \rangle = \epsilon \langle P_0 \rangle$; ϵ = FRACTION OF POWER ABSORBED

THEN $$\epsilon = -\frac{4\pi}{c} \frac{Z}{a} \int \bar{J} \cdot \bar{E}_{0t}$$

ALSO $$\langle P_{TRANS} \rangle = (1-\epsilon)\langle P_0 \rangle = \frac{c}{8\pi} \frac{(1-\epsilon)a^2}{Z}$$

IF $\epsilon \ll 1$ THE STRENGTH OF THE TRANSMITTED FIELD IS

$$\bar{E}_{TRANS} = \sqrt{1-\epsilon}\, a\bar{E}_0 \approx \left(1 - \frac{\epsilon}{2}\right) a\bar{E}_0$$

THE TIME REVERSAL CONCLUSION IS THAT

$$\bar{E}_{RADIATED} + \bar{E}_{TRANS} = a\bar{E}_0 \quad (= \bar{E}_{INCIDENT})$$

SO $\bar{E}_{RADIATED} = \frac{\epsilon}{2} a \bar{E}_0$ IS OUR DESIRED FIELD

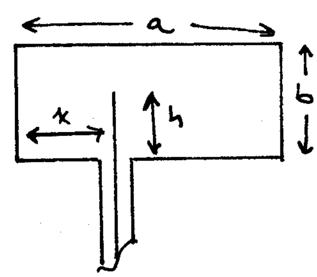
$$\underline{\bar{E}_{RADIATED} = -\frac{2\pi}{c} Z \bar{E}_0 \int \bar{J} \cdot \bar{E}_{0t}}$$

WHERE $\bar{E}_{0t}$ FLOWS <u>TOWARDS</u> THE SOURCE.

I ADMIT THAT THE DERIVATION SEEMS SOMEWHAT BIZARRE, BUT IT SHOWS THE POWER OF THE INVARIANCE ARGUMENTS

<u>EXAMPLE</u>. SUPPOSE THE CURRENT DISTRIBUTION IN THE WIRE IS

$$I_y = I_0 \sin\left[\frac{\omega}{c}(h-y)\right] e^{-i\omega t}$$



WHICH AT LEAST VANISHES AT THE TOP OF THE WIRE.

ω IS CHOSEN SO ONLY THE LOWEST MODE, THE TE_{10} MODE CAN PROPAGATE.

IN THE TE_{10} MODE, $K_g = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$

AND $E_x = 0 \qquad E_z = 0 \qquad E_y = E_{oy} \sin\frac{\pi x}{a}$

WE DEFINE THE NORMALIZED WAVE $\bar{E}_o$ BY $\int E_y^2 \, dy\, dx = 1$

SO $E_{oy}^2 \frac{ab}{2} = 1$ OR $E_{oy} = \sqrt{\frac{2}{ab}}$

THEN
$$\int \bar{j}\cdot\bar{E}_o = \int_0^y dy \cdot I_o \sin\frac{\omega}{c}(h-y)\sqrt{\frac{2}{ab}} \sin\frac{\pi x}{a}$$
$$= -\frac{c I_o}{\omega}\sqrt{\frac{2}{ab}} \sin\frac{\pi x}{a}\left(1-\cos\frac{\omega h}{c}\right)$$

AND SO
$$\bar{E}_{RAD} = 2\pi \frac{I_o}{\omega}\sqrt{\frac{2}{ab}}\frac{K}{K_g}\sin\frac{\pi x}{a}\left(1-\cos\frac{\omega h}{c}\right)\bar{E}_o$$

$2\sin^2\frac{\omega h}{2c}$

AND THE POWER DELIVERED INTO THE TE_{10} MODE

IS
$$\langle P\rangle_{10} = \frac{c}{8\pi Z}\int E_{RAD}^2 \, dAREA$$
$$= \frac{c}{8\pi}\frac{K_g}{K} 4\pi^2 \frac{I_o^2}{\omega^2}\frac{2}{ab}\frac{K^2}{K_g^2}\sin^2\frac{\pi x}{a} \cdot 4\sin^4\frac{\omega h}{2c}$$
$$= \frac{4\pi I_o^2}{ab\omega K_g}\sin^2\frac{\pi x}{a}\sin^4\frac{\omega h}{2c}$$

FOR WHAT IT'S WORTH.

NOTE THAT OUR APPROACH DEPENDS ON THE CURRENT DISTRIBUTION BEING KNOWN. IT IS NOT CLEAR THAT ANY ARBITRARY DISTRIBUTION SPECIFIED BY US CAN ACTUALLY BE ACHIEVED IN PRACTICE...

A FINAL REMARK: WAVES INSIDE PIPES CAN ONLY EXIST IN THE FORM OF A COUNTABLE NUMBER OF MODES. A GENERAL WAVE IS THEN A SUPERPOSITION OF SEVERAL MODES:

$$\bar{E} = \sum_{ij} a_{ij}\bar{E}_{ij}$$

WHERE i & j ARE MODE INDICES.

THIS IS A KIND OF FOURIER SERIES EXPANSION! IT TURNS OUT THAT THE MODES ARE INDEED 'COMPLETE' AND 'ORTHOGONAL' IN SUCH A WAY AS TO MAKE THIS EXPANSION A WELL-DEFINED AND USEFUL CONCEPT...