Special Relativity

[This lecture roughly parallels Becker secs. 73-84, and 89-90]

Newton's 1st law embodies the notion of Galilean relativity: the laws of physics should appear the same in any inertial frame.

An inertial frame is a coordinate system in which Newton's 1st law holds: bodies at rest or in a state of uniform motion remain so unless acted upon by an outside force. Different inertial frames can be distinguished by their relative velocities.

Maxwell added the complication that the velocity of light can be determined by performing electro- and magneto static experiments, and so this velocity is expected, by the principle of relativity, to be the same in all inertial frames.

Einstein showed how to reconcile the apparent contradictions in the latter statement - by postulating that the speed of light is the maximum velocity with which any signal can be sent. This postulate, strictly speaking, has nothing to do with electricty and magnetism, and allows the results of Maxwell's equations to be understood in a broader context.

Introductory discussions of special relativity spend a lot of time with paradoxes as to apparent differences in measurements made by different observers. We presume you are familiar with the resolution of the paradoxes - which seem to contradict the very idea of relativity.

Instead we wish to develop a notation which allows the laws of physics to be expressed independently of a particular observer. Also, if a particular observer expresses a law in a manner which is not obviously 'relativistic', we wish to show how to transform this law to the special viewpoint of another observer.

The Lorentz Transformation

An event is a moment in time at some particular point in space. In some inertial frame, coordinates t and $\bar{x}$ label this event. In a different inertial frame, moving at velocity $\bar{v}$ relative to the first frame (according to an observer in the first frame), the coordinates of the event are

$$t' = \gamma\left(t - \frac{\bar{v}\cdot\bar{x}}{c^2}\right) \qquad \text{with} \qquad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1$$

$$\bar{x}_{\parallel}' = \gamma(\bar{x}_{\parallel} - \bar{v}t) \qquad \text{and} \qquad \bar{x}_{\parallel} = (\bar{x}\cdot\hat{v})\hat{v}$$

$$\bar{x}_{\perp}' = \bar{x}_{\perp} \qquad\qquad \bar{x}_{\perp} = \bar{x} - \bar{x}_{\parallel}$$

WE ALSO DEFINE $\bar{\beta} \equiv \bar{v}/c$. THEN

$$ct' = \gamma (ct - \bar{\beta} \cdot \bar{x})$$
$$\bar{x}' = \bar{x} + (\gamma - 1)(\bar{x} \cdot \hat{\beta})\hat{\beta} - \gamma \bar{\beta} ct$$

THESE ARE 2 VERSIONS OF THE FAMOUS LORENTZ TRANSFORMATION

[APPARENTLY THIS WAS FIRST WRITTEN DOWN IN 1887 BY W. VOIGHT, WHO NOTED THAT THE ELECTROMAGNETIC WAVE EQUATION RETAINED ITS FORM UNDER THIS TRANSFORMATION $\Leftrightarrow$ c IS INDEPENDENT OF THE FRAME]

THE POSITION 4-VECTOR

WE INTRODUCE A VERY USEFUL NOTATION:

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, \bar{x}) = \text{POSITION 4-VECTOR}$$

WE WILL USE GREEK INDICES AS SUBSCRIPTS OF 4-VECTORS
LATIN INDICES AS SUBSCRIPTS OF 3-VECTORS.

IN THIS NOTATION THE LORENTZ TRANSFORMATION BECOMES

$$\underline{x'_\mu = L_{\mu\nu}(\bar{\beta})\, x_\nu} \qquad \text{[SUMMATION CONVENTION]}$$

WHERE
$$L_{\mu\nu} = \left(\begin{array}{c|c} \gamma & -\gamma\bar{\beta} \\ \hline -\gamma\bar{\beta} & \delta_{ij} + (\gamma-1)\hat{\beta}_i\hat{\beta}_j \end{array}\right)$$

THE INVERSE TRANSFORMATION IS $\underline{x_\mu = L_{\mu\nu}(-\bar{\beta})\, x'_\nu}$

THE LORENTZ CONTRACTION

[FITZGERALD 1889 - IN A PAPER WITH NO EQUATIONS
LORENTZ 1892 - CORRECT TO ORDER $(v/c)^2$]

AN OBJECT AT REST IN FRAME S' EXTENDS FROM $x' = 0$ TO $x' = L'$

FRAME S' MOVES WITH VELOCITY $\bar{v} = (v, 0, 0)$ RELATIVE TO FRAME S.
AT $t = 0$ IN FRAME S THE OBJECT IS OBSERVED TO EXTEND FROM x_{LEFT} TO x_{RIGHT}. WHAT IS $L = x_{RIGHT} - x_{LEFT}$ IN FRAME S?

IT IS EASIEST TO USE $x' = \gamma(x - vt)$

SO
$$0 = \gamma\, x_{LEFT}$$
$$L' = \gamma\, x_{RIGHT}$$

[SINCE BOTH OBSERVATIONS OF x_L & x_R ARE MADE AT THE SAME TIME t IN FRAME S]

$$\Rightarrow \underline{\underline{L = \frac{L'}{\gamma} < L'}}$$

RELATIVITY OF SIMULTANEITY

AN OBSERVER IN FRAME S' IS NOT COMPLETELY SURPRISED BY THIS RESULT, BECAUSE HE THINKS THE MEASUREMENTS OF X_{LEFT} AND X_{RIGHT} WERE MADE AT TWO DIFFERENT TIMES. [RELATIVITY OF SIMULTANEITY]

IN GENERAL $t' = \gamma\left(t - \frac{vx}{c}\right)$ SO $t'_{LEFT} = 0$ BUT $t'_{RIGHT} = -\frac{v\gamma}{c^2} x_{RIGHT}$

[SUPPOSING $t=0$ = TIME OF OBSERVATION IN FRAME S] $= -\frac{v}{c^2}L' < 0$

BUT DURING TIME $\Delta t' = \frac{v}{c^2}L'$, FRAME S MOVES BY $\Delta L' = \frac{v^2}{c^2}L'$

ACCORDING TO THE OBSERVER IN FRAME S'. HENCE, EVEN IF THERE WERE NO LORENTZ CONTRACTION, THE OBSERVER IN FRAME S' WOULD EXPECT THE MEASUREMENT IN FRAME S TO YIELD $L'' = L' - \Delta L' = (1 - v^2/c^2)L' = \frac{L'}{\gamma^2}$

FINALLY, WE INVOKE THE RELATIVITY OF THE LORENTZ CONTRACTION: IF OBSERVER S' MEASURES AS L'' A LENGTH WHICH OBSERVER S CALLS L, THEN WE MUST HAVE $L'' = L/\gamma$ OR $L = \gamma L'' = L'/\gamma$.

TIME DILATION

WE (IN FRAME S) WATCH A CLOCK WHICH IS AT REST IN THE MOVING FRAME S', WHILE THAT CLOCK CHANGES ITS READING FROM $t'=0$ TO $t'=T'$. SUPPOSE THE MOVING CLOCK IS AT $x'=0$.

IN GENERAL $t = \gamma\left(t' + \frac{v}{c^2}x'\right)$ (THE INVERSE TRANSFORMATION)

SO $t=0$ WHEN $x'=0$ & $t'=0$

WHILE $t = \gamma T'$ WHEN $x'=0$ & $t'=T'$

HENCE $T = \gamma T' > T'$ IS THE TIME INTERVAL IN OUR FRAME

WE THINK THE CLOCK IN THE MOVING FRAME RUNS SLOW.

THE INVARIANT INTERVAL - PROPER TIME

WITH LENGTHS AND TIMES APPARENTLY CHANGING FROM FRAME TO FRAME, IS THERE ANYTHING WHICH REMAINS THE SAME?

CONSIDER THE CONTRACTION CASE.

WE SAW ABOVE THAT $t'_{RIGHT} - t'_{LEFT} = \Delta t' = -\frac{v}{c^2}L'$

HENCE $(ct'_R - ct'_L)^2 - (x'_R - x'_L)^2 = \beta^2 L'^2 - L'^2 = (\beta^2 - 1)L'^2$

BUT $(ct_R - ct_L)^2 - (x_R - x_L)^2 = 0 - \frac{L'^2}{\gamma^2} = (\beta^2 - 1)L'^2$ ALSO!

CONSIDER THE TIME DILATION CASE.

USING $x = \gamma(x' + vt')$ WE HAVE $X_{INITIAL} = \gamma(0+0) = 0$
$X_{FINAL} = \gamma(0 + vT') = \gamma v T'$

SO $(ct_f - ct_i)^2 - (x_f - x_i)^2 = c^2\gamma^2 T'^2 - \gamma^2 v^2 T'^2 = c^2 T'^2$

WHILE $(ct_f' - ct_i')^2 - (x_f' - x_i')^2 = c^2 T'^2 - 0 = c^2 T'^2$ ALSO!

WE ARE LED TO THE CONCLUSION THAT THE QUANTITY

$$s^2 = (ct)^2 - \bar{x}^2 = x_0^2 - \bar{x}^2$$

IS INVARIANT UNDER A LORENTZ TRANSFORMATION.
(WE CALL ANY SCALAR INVARIANT A LORENTZ SCALAR)

THE INTERVAL $ds^2 = (c\,dt)^2 - (d\bar{x})^2$ IS ALSO INVARIANT.

$$= (c\,dt)^2\left(1 - \frac{1}{c^2}\left(\frac{d\bar{x}}{dt}\right)^2\right) = (c\,dt)^2\left(1 - \frac{v^2}{c^2}\right) = \left(\frac{c\,dt}{\gamma}\right)^2$$

HENCE THE QUANTITY $d\tau = \frac{dt}{\gamma}$ ≡ PROPER TIME $[ds = c\,d\tau]$

IS AN INVARIANT MEASURE OF THE 'INTERVAL' BETWEEN TWO EVENTS IN THE HISTORY OF A MOVING OBJECT.
IN WRITING FRAME-INDEPENDENT LAWS WHICH RELATE DIFFERENT EVENTS, IT WILL BE APPROPRIATE TO USE $d\tau$ RATHER THAN QUANTITIES dt OR $d\bar{x}$ WHICH ARE SUBJECT TO THE CONFUSING DILATIONS AND CONTRACTIONS.

THE SCALAR PRODUCT

IF a_μ IS ANY 4-VECTOR THEN $a_0^2 - \bar{a}^2$ IS A LORENTZ INVARIANT. THIS MAY BE FURTHER GENERALISED: IF a_μ AND b_μ ARE ANY TWO 4-VECTORS THEN

$$a \cdot b \equiv a_0 b_0 - \bar{a}\cdot\bar{b}$$ IS A LORENTZ INVARIANT.

WE THEREFORE CAN DEFINE THIS AS THE SCALAR PRODUCT

[THERE IS NO USEFUL DEFINITION OF A VECTOR PRODUCT - A CONCEPT WHICH IS PECULIAR TO 3-DIMENSIONS...]

IT IS USEFUL TO INTRODUCE A SECOND SUMMATION CONVENTION,

$$a \cdot b = a_\mu b^\mu = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3$$

AS OPPOSED TO

$$a_\mu b_\mu = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$

THE WAVE 4-VECTOR

SOME COMPONENT OF A PLANE WAVE MIGHT BE DESCRIBED BY

$$\psi = \psi_0 \, e^{i(\bar{k}\cdot\bar{x} - \omega t)}$$

OBSERVERS IN DIFFERENT FRAMES WILL ALWAYS AGREE THAT SOME PARTICULAR "EVENT" x_μ IS ON A CREST OR A TROUGH.

HENCE THE PHASE φ OF THE WAVE, $\bar{k}\cdot\bar{x} - \omega t$, MUST BE A LORENTZ SCALAR.

WE RECOGNIZE THIS AS THE SCALAR PRODUCT OF TWO 4-VECTORS:

$$\varphi = -k_\mu x^\mu$$

WHERE $\underline{k_\mu} = \left(\frac{\omega}{c}, \bar{k}\right) = \underline{(k, \bar{k})}$ = WAVE NUMBER 4-VECTOR

IF k_μ IS A 4-VECTOR, THEN ITS COMPONENTS IN DIFFERENT FRAMES ARE RELATED BY THE LORENTZ TRANSFORMATION:

$$k_\mu' = L_{\mu\nu}(\bar{\beta})\, k_\nu$$

$$\Rightarrow \quad k' = \gamma(k + \bar{\beta}\cdot\bar{k})$$

$$\bar{k}' = \bar{k} + (\gamma - 1)(\bar{k}\cdot\hat{\beta})\hat{\beta} + \gamma\bar{\beta} k$$

IN PARTICULAR, $\omega' = k'c = \gamma(kc + \bar{\beta}\cdot\bar{k}c)$

OR $\omega' = \omega \dfrac{(1 + \hat{n}\cdot\bar{\beta})}{\sqrt{1 - v^2/c^2}}$ THE RELATIVISTIC DOPPLER SHIFT!

THE QUANTITY $k_\mu k^\mu$ IS A LORENTZ SCALAR, BUT $k_\mu k^\mu = k^2 - \bar{k}^2 = 0$

HENCE THERE IS NO FRAME IN WHICH $k \neq 0$ WHILE $\bar{k} = 0$.

THAT IS, THERE IS NO FRAME IN WHICH A TRAVELLING LIGHT WAVE APPEARS TO BE AT REST! THE SPEED OF LIGHT, c, IS THE SAME IN ALL INERTIAL FRAMES.

SPACE-LIKE, TIME-LIKE AND LIGHT-LIKE 4-VECTORS

ANY 4 VECTOR a_μ FOR WHICH $a_\mu a^\mu = 0$ IS CALLED LIGHT-LIKE

BOTH a_0 AND $\bar{a}$ ARE NON-ZERO IN ALL INERTIAL FRAMES

IF $a_\mu a^\mu = a^2 > 0$ WE SAY a_μ IS TIME-LIKE

THERE IS A SPECIAL FRAME IN WHICH $a_\mu = (a, \bar{0})$ = PURE TIME COMPONENT

IF $a_\mu a^\mu = -a^2 < 0$ WE SAY a_μ IS SPACE-LIKE

THERE IS A SPECIAL FRAME IN WHICH $a_\mu = (0, \bar{a})$ WITH $|\bar{a}| = a$.

IN THIS FRAME a_μ IS A PURE SPATIAL 3-VECTOR.

THE VELOCITY 4-VECTOR

A PLAN IS NOW EMERGING. WE TRY TO CONVERT ALL RELEVANT QUANTITIES INTO LORENTZ SCALARS, 4-VECTORS, 4-TENSORS .....

A GOOD EXAMPLE IS THE VELOCITY OF PARTICLE $\bar{v}$, FOR WHICH $|\bar{v}| < c$. THE PARTICLE CAN CERTAINLY BE AT REST, AND IF SO THE VELOCITY 4-VECTOR MUST HAVE THE FORM (IN THE REST FRAME)

$$u_\mu = (a, \bar{0}) \qquad \text{(TIME-LIKE)}$$

TO DETERMINE THE CONSTANT a, WE TRANSFORM TO A FRAME IN WHICH THE PARTICLE HAS VELOCITY $\bar{v}$.

$$\Rightarrow \begin{cases} u_0' = \gamma a \\ \bar{u}' = \gamma \bar{\beta} a \end{cases} \quad \text{OR} \quad u_\mu' = \left(\gamma a, \gamma \frac{\bar{v}}{c} a\right)$$

WE COULD CHOOSE $a = \frac{c}{\gamma}$ $\Rightarrow$ $u_\mu' = (c, \bar{v})$

WHICH LOOKS NICE. BUT NOTE THAT IN THE 'REST FRAME' WE WOULD THEN HAVE $u_\mu = \left(\frac{c}{\gamma}, \bar{0}\right)$ WHERE $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ AND v REFERS TO THE VELOCITY OF SOME ARBITRARY FRAME. THIS IS NOT VERY SATISFACTORY.

A MUCH MORE REASONABLE CHOICE IS $a = 1$ (OR MAYBE $a = c$)

WE DEFINE $u_\mu = \underline{\left(\gamma, \gamma \frac{\bar{v}}{c}\right)} = \underline{(\gamma, \gamma\bar{\beta})}$ = <u>VELOCITY 4-VECTOR</u>

WE CAN EASILY DERIVE THE FORMULA FOR THE ADDITION OF VELOCITIES: A PARTICLE HAS VELOCITY $\bar{w}$ IN FRAME S', WHICH FRAME HAS VELOCITY $\bar{v}$ RELATIVE TO OUR FRAME. WHAT IS THE PARTICLE'S VELOCITY IN OUR FRAME?

THE 4-VELOCITY IS $u_\mu' = (\gamma', \gamma'\bar{\beta}')$ IN FRAME S'

WHERE $\bar{\beta}' = \bar{w}/c$ AND $\gamma' = 1/\sqrt{1 - \beta'^2}$

IN OUR FRAME, WE OBTAIN BY A LORENTZ TRANSFORMATION

$$\bar{u} = \bar{u}' + (\gamma - 1)(\bar{u}' \cdot \hat{\beta})\hat{\beta} + \gamma\bar{\beta} u_0' \qquad \text{WHERE } \bar{\beta} = \frac{\bar{v}}{c} \text{ AND } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$u_0 = \gamma(u_0' + \bar{\beta}\cdot\bar{u}') = \gamma\gamma'(1 + \bar{\beta}\cdot\bar{\beta}')$$

NOW $\bar{u} = u_0 \cdot \frac{\text{VELOCITY IN OUR FRAME}}{c}$

SO $$\frac{\overline{\text{VELOCITY}}}{c} = \frac{\gamma'\left[\bar{\beta}' + (\gamma - 1)(\bar{\beta}'\cdot\hat{\beta})\hat{\beta} + \gamma\bar{\beta}\right]}{\gamma\gamma'(1 + \bar{\beta}\cdot\bar{\beta}')}$$

THE COMPONENT OF VELOCITY PARALLEL TO $\bar{v}$ IS $\frac{\overline{\text{VEL}}}{c}\cdot\hat{\beta} = \frac{\bar{\beta}'\cdot\hat{\beta} + \beta}{1 + \bar{\beta}\cdot\bar{\beta}'} = \frac{\beta'_{\parallel} + \beta}{1 + \bar{\beta}\cdot\bar{\beta}'}$

THE TRANSVERSE COMPONENT OF THE VELOCITY IS

$$\frac{\text{VEL}_\perp}{c} = \frac{\overline{\text{VEL}}}{c} - \frac{\overline{\text{VEL}}_\parallel}{c} = \frac{\beta'_\perp}{\gamma(1+\bar{\beta}\cdot\bar{\beta}')}$$ AFTER A LINE OR TWO.

ENERGY-MOMENTUM 4-VECTOR

SUPPOSE A PARTICLE HAS MASS M_0 WHEN AT REST. THEN WHEN LOOKING FOR THE VELOCITY 4-VECTOR WE MIGHT HAVE CHOSEN TO SET THE CONSTANT $a = M_0$

THEN $(M_0, \bar{0})$ IS A CANDIDATE FOR A 4-VECTOR.

IN A FRAME IN WHICH THE PARTICLE HAS VELOCITY $\bar{v}$, THIS 4-VECTOR HAS COMPONENTS

$$\left(M_0\gamma,\ M_0\gamma\frac{\bar{v}}{c}\right) = \left(\frac{E}{c^2},\ \frac{\bar{P}}{c}\right)$$

WHERE $E = M_0\gamma c^2$ = ENERGY $\quad \bar{P} = M_0\gamma\bar{v}$ = MOMENTUM.

WE DEFINE $P_\mu = (E, \bar{P}c)$ = ENERGY-MOMENTUM 4-VECTOR

$$= (M_0\gamma c^2,\ M_0\gamma\bar{\beta}c^2) = (Mc^2,\ M\bar{\beta}c^2) = M_0c^2 u_\mu$$

WHERE $M = M_0\gamma$ = MASS OF THE MOVING PARTICLE.

ANGULAR MOMENTUM 4-TENSOR

IN 3-SPACE WE WRITE $\bar{M} = \bar{r}\times\bar{P}$ AS THE ANGULAR MOMENTUM (WE RESERVE THE LETTER L FOR THE LORENTZ TRANSFORMATION)

IN COMPONENTS $\quad M_i = r_j P_k - r_k P_j \quad (i\,j\,k)$ = CYCLIC PERM OF (123)

IN 4-DIMENSIONS THE BEST WE CAN DO IS TO DEFINE A TENSOR

$$M_{\mu\nu} = \frac{x_\mu P_\nu}{c} - \frac{x_\nu P_\mu}{c}$$

THIS TENSOR IS ANTI-SYMMETRIC $\Rightarrow$ ONLY 6 INDEPENDENT COMPONENTS.

$$M_{\mu\nu} = \begin{pmatrix} 0 & M_{01} & M_{02} & M_{03} \\ -M_{01} & 0 & M_{12} & M_{13} \\ -M_{02} & -M_{12} & 0 & M_{23} \\ -M_{03} & -M_{13} & -M_{23} & 0 \end{pmatrix}$$

$M_{\mu\nu}$ IS SOMETIMES CALLED A 6-VECTOR, BEING COMPOSED OF 2 3-VECTORS $\bar{M}$ AND $\bar{M}'$

WE CAN IDENTIFY 2 ENTITIES WHICH LOOK LIKE 3-VECTORS:

$$\bar{M} = (M_{23},\ M_{13},\ M_{12}) \quad \text{AND} \quad \bar{M}' = (M_{01},\ M_{02},\ M_{03})$$

SO $\bar{M} = \bar{r}\times\bar{P}$ BUT WHAT IS $\bar{M}'$??

NOTE THAT IF $\bar{F}_{EXT} = 0$, P_μ = CONST $\Rightarrow$ BOTH $\bar{M}$ AND $\bar{M}'$ ARE CONSERVED.

THE LORENTZ TRANSFORMATION OF A TENSOR IS

$$M'_{\mu\nu} = L_{\mu\alpha} L_{\nu\beta} M_{\alpha\beta}$$

THE CHARGE-CURRENT 4-VECTOR

WE CONSIDER A CHARGE DENSITY ρ_0 WHICH IS AT REST.
IF WE TAKE $a = \rho_0$ AS THE TIME COMPONENT OF A 4-VECTOR, THEN IN A GENERAL FRAME (IN WHICH THE CHARGE HAS VELOCITY $\bar{v}$) WE HAVE

$$\underline{J_\mu} = \left(\gamma\rho_0 , \gamma \frac{\rho_0 \bar{v}}{c}\right) = \underline{\left(\rho , \frac{\bar{J}}{c}\right)} = \text{CHARGE-CURRENT 4-VECTOR}$$

WHERE $\rho \equiv \gamma\rho_0$ AND $\bar{J} = \rho\bar{v}$

NOTE THAT $\rho > \rho_0$ BECAUSE OF THE LORENTZ CONTRACTION.
CHARGE IS CONSIDERED TO BE A LORENTZ SCALAR QUANTITY!

DERIVATIVES

SUPPOSE $f(x_\mu)$ IS A LORENTZ SCALAR QUANTITY.

THEN $df = \frac{\partial f}{\partial x_0} dx_0 + \frac{\partial f}{\partial x_i} dx_i$ IS ALSO.

NOW dx_μ IS A 4-VECTOR (BEING THE DIFFERENCE BETWEEN 2 4-VECTORS)

HENCE WE CAN DEFINE A 4-VECTOR $\frac{\partial f}{\partial x_\mu}$ SO THAT $df = \frac{\partial f}{\partial x_\mu} dx^\mu$ = SCALAR

NOTE THE WE MUST SET $\frac{\partial f}{\partial x_\mu} = \left(\frac{\partial f}{\partial x_0} , -\frac{\partial f}{\partial \bar{x}}\right) = \left(\frac{1}{c}\frac{\partial f}{\partial t} , -\bar{\nabla} f\right)$

MINUS SIGN NEEDED TO SATISFY THE CHAIN RULE AND THE SUMMATION CONVENTION.

WE ALSO DEFINE $\underline{\partial_\mu} = \underline{\left(\frac{1}{c}\frac{\partial}{\partial t} , -\bar{\nabla}\right)}$ = 4-DERIVATIVE OPERATOR

THEN IF f_μ IS A 4-VECTOR $\partial_\mu f^\mu$ IS A LORENTZ SCALAR

$f_{\mu\nu}$ IS A 4-TENSOR $\partial^\mu f_{\mu\nu}$ IS A 4-VECTOR

$\partial^\mu \partial^\nu f_{\mu\nu}$ IS A SCALAR ....

Ph 206 LECTURE 18

THE CONTINUITY EQUATION $\quad \bar{\nabla}\cdot\bar{J} + \frac{\partial \rho}{\partial t} = 0$

IN 4-VECTOR NOTATION: $\partial_\mu J^\mu = \frac{1}{c}\frac{\partial \rho}{\partial t} + \frac{\bar{\nabla}\cdot\bar{J}}{c} = 0$

[THIS SUGGESTS THAT MAXWELL'S INSISTENCE THAT $\bar{\nabla}\cdot\bar{J}_{TOT} = 0$ WAS RELATIVISTICALLY MISGUIDED!]

THE D'ALEMBERTIAN OPERATOR

THE SECOND DERIVATIVE OPERATOR $\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ IS A LORENTZ INVARIANT OPERATOR. WE DEFINE $\Box = \partial_\mu \partial^\mu$ = D'ALEMBERTIAN

THE 4-POTENTIAL

THE WAVE EQUATIONS FOR THE ELECTROMAGNETIC POTENTIALS ARE

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho \qquad \nabla^2\bar{A} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -4\pi\frac{\bar{J}}{c}$$

IN 4-VECTOR NOTATION $-\Box(\phi, \bar{A}) = -4\pi\left(\rho, \frac{\bar{J}}{c}\right) = -4\pi J_\mu$

WE DEFINE ϕ_μ ($= A_\mu$ SOMETIMES) $= (\phi, \bar{A})$ = 4-POTENTIAL

THEN $\Box\phi_\mu = 4\pi J_\mu$ IS THE WAVE EQUATION.

THE LORENTZ GAUGE CONDITION WAS ASSUMED IN WRITING DOWN THE WAVE EQUATIONS: $\bar{\nabla}\cdot\bar{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$ (p 174)

IN 4-VECTOR NOTATION THIS IS JUST $\partial_\mu \phi^\mu = 0$.

THE FIELDS $\bar{E}$ AND $\bar{B}$ THE RELATIONS $\bar{B} = \bar{\nabla}\times\bar{A}$, $\bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$

CANNOT BE DIRECTLY TRANSCRIBED INTO 4-VECTOR FORM.

BUT NOTE THAT $E_i = -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial ct} = \partial_i\phi_0 - \partial_0\phi_i \qquad \left(\partial_i = -\frac{\partial}{\partial x_i}\right)$

WHILE $B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} = -(\partial_j\phi_k - \partial_k\phi_j)$

THIS SUGGESTS WE DEFINE A TENSOR $F_{\mu\nu} = \partial_\mu\phi_\nu - \partial_\nu\phi_\mu$

$$F_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} = \underline{\text{FIELD}}\ \underline{\text{TENSOR}}$$

THIS ANTI-SYMMETRIC TENSOR IS ANOTHER EXAMPLE OF A 'SIX-VECTOR'.

<u>MAXWELL'S</u> <u>EQUATIONS</u> THESE INVOLVE FIRST DERIVATIVES OF THE FIELDS. SO WE LOOK AT

$\partial^\mu F_{\mu\nu}$. FOR EXAMPLE, TAKE $\nu = 0$.

THEN $\partial^\mu F_{\mu 0} = -\left(-\frac{\partial}{\partial x_i}\right) F_{i0} = \frac{\partial E_i}{\partial x_i} = \bar{\nabla}\cdot\bar{E} = 4\pi\rho = 4\pi j_0$

$\nu = 1 \Rightarrow \partial^\mu F_{\mu 1} = \frac{1}{c}\frac{\partial F_{01}}{\partial t} - \left(-\frac{\partial}{\partial x_2}\right)F_{21} - \left(-\frac{\partial}{\partial x_3}\right)F_{23} = -\frac{1}{c}\frac{\partial E_1}{\partial t} + \bar{\nabla}\times\bar{B}\big|_1 = \frac{4\pi}{c}\bar{J}_1 = 4\pi j_1$

HENCE $\underline{\partial^\mu F_{\mu\nu} = 4\pi j_\nu} \iff \begin{cases} \bar{\nabla}\cdot\bar{E} = 4\pi\rho \\ \bar{\nabla}\times B - \frac{1}{c}\frac{\partial \bar{E}}{\partial t} = \frac{4\pi \bar{J}}{c} \end{cases}$

BUT WHERE ARE THE OTHER TWO EQUATIONS?

ONE WAY IS TO INTRODUCE THE <u>DUAL</u> TENSOR

$$G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

THEN $\underline{\partial^\mu G_{\mu\nu} = 0} \iff \begin{cases} \bar{\nabla}\cdot\bar{E} = 0 \\ \bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial\bar{B}}{\partial t} = 0 \end{cases}$

THESE COULD ALSO HAVE BEEN WRITTEN $\underline{\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0}$

FOR WHAT IT'S WORTH.

THE QUANTITIES $-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}G_{\mu\nu}G^{\mu\nu} = E^2 - B^2$

AND $\frac{1}{4}F_{\mu\nu}G^{\mu\nu} = \bar{E}\cdot\bar{B}$

ARE LORENTZ INVARIANTS OF THE FIELDS. THESE ARE THE ONLY SUCH INVARIANTS.
OF COURSE $\phi_\mu\phi^\mu = \phi^2 - \bar{A}^2$ IS AN INVARIANT ALSO.

TRANSFORMATION OF THE FIELDS

$$F'_{\mu\nu} = L^{(\bar\beta)}_{\mu\alpha} L^{(\bar\beta)}_{\nu\beta} F_{\alpha\beta}$$

$$\Rightarrow \begin{cases} \bar E' = \gamma\left(\bar E + \frac{\bar v}{c}\times\bar B\right) - (\gamma-1)(\bar E\cdot\hat v)\,\hat v \\ \bar B' = \gamma\left(\bar B - \frac{\bar v}{c}\times\bar E\right) - (\gamma-1)(\bar B\cdot\hat v)\,\hat v \end{cases}$$

$$\text{OR} \begin{cases} \bar E'_{\parallel} = \bar E_{\parallel} \qquad \bar E'_{\perp} = \gamma\left(\bar E_{\perp} + \frac{\bar v}{c}\times\bar B\right) \\ \bar B'_{\parallel} = \bar B_{\parallel} \qquad \bar B'_{\perp} = \gamma\left(\bar B_{\perp} - \frac{\bar v}{c}\times\bar E\right) \end{cases}$$

DERIVATIVES WITH RESPECT TO PROPER TIME

AS INTRODUCED ON P 214 $d\tau = \frac{ds}{c} = \frac{1}{c}\sqrt{dx_\mu dx^\mu} = \frac{dt}{\gamma}$

IS A LORENTZ INVARIANT.

HENCE $\frac{df}{d\tau}$, $\frac{df_\mu}{d\tau}$, $\frac{df_{\mu\nu}}{d\tau}$ ARE LORENTZ SCALARS, VECTORS OR TENSORS

IF f, f_μ $f_{\mu\nu}$ ARE SCALARS, VECTORS, OR TENSORS

4-VELOCITY AND ACCELERATION

CONSIDER $x_\mu = (ct, \bar x)$

$$\frac{dx_\mu}{ds} = \left(\gamma, \gamma\frac{\dot{\bar x}}{c}\right) = (\gamma, \gamma\bar\beta) = u_\mu = \text{4-VELOCITY}$$

(SOME PEOPLE DEFINE $u_\mu = (c\gamma, \gamma\bar v)$. THEN $u_\mu = \frac{dx_\mu}{d\tau}$)

WE DEFINE $a_\mu = \frac{du_\mu}{ds} = \left(\frac{\gamma}{c}\frac{d\gamma}{dt}, \frac{\gamma}{c}\frac{d\gamma\bar\beta}{dt}\right)$ = 4-ACCELERATION

IT MAY BE WORTH NOTING THAT $\frac{d\gamma}{dt} = \gamma^3\bar\beta\cdot\frac{d\bar\beta}{dt}$

$$\text{SO} \quad a_\mu = \left(\frac{\gamma^4\bar\beta}{c}\cdot\frac{d\bar\beta}{dt}, \frac{\gamma^4\bar\beta}{c}\left(\bar\beta\cdot\frac{d\bar\beta}{dt}\right) + \frac{\gamma^2}{c}\frac{d\bar\beta}{dt}\right)$$

IN THE PARTICLE'S REST FRAME THIS REDUCES TO $a_\mu = \left(0, \frac{1}{c}\frac{d\bar\beta}{dt}\right) = \left(0, \frac{\bar a}{c^2}\right)$

WHERE $\bar a = \ddot{\bar x}$.

CAN YOU SHOW THAT $a_\mu a^\mu = -\frac{\gamma^6}{c^2}\left[\dot{\bar\beta}^2 - (\bar\beta\times\dot{\bar\beta})^2\right]$?

$$a^{*2} = \gamma^6 c^2\left[\dot\beta^2 - (\beta\times\dot\beta)^2\right]$$

MINKOWSKI FORCE

WE WISH TO GENERALISE $\bar{F} = m\bar{a} = \frac{d\bar{p}}{dt}$

SUPPOSE WE WRITE $f_\mu = \frac{dP_\mu}{ds} = \left(\frac{\gamma}{c}\frac{dE}{dt},\ \gamma\frac{d\bar{p}}{dt}\right)$ $\quad \left[P_\mu = (E, \bar{p}c)\right]$

SO $f_\mu = (\gamma\bar{F}\cdot\bar{\beta},\ \gamma\bar{F}) \equiv$ MINKOWSKI FORCE

WHERE $\bar{F}$ = NEWTONIAN FORCE, $\bar{F}\cdot\bar{\beta} = \frac{\bar{F}\cdot\bar{v}}{c} = \frac{\text{POWER}}{c}$

THE LORENTZ FORCE

$$f_\mu = F_{\mu\nu}j^\nu = \left(\frac{\bar{J}\cdot\bar{E}}{c},\ \rho\bar{E} + \frac{\bar{J}}{c}\times\bar{B}\right)$$

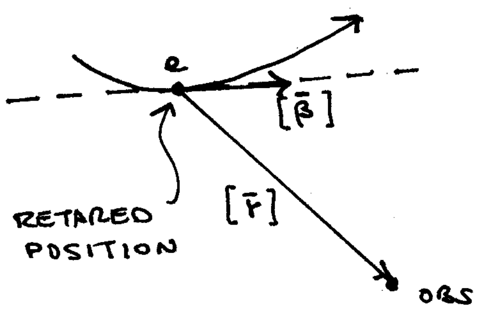
FOR A SINGLE CHARGE e MOVING WITH VELOCITY $\bar{\beta}$, $\rho \to e\gamma$ AND

$$f_\mu = \left(e\gamma\bar{\beta}\cdot\bar{E},\ e\gamma\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right)\right) = \frac{dP_\mu}{ds} = m_0c^2\frac{du_\mu}{ds} = m_0\gamma c\left(\frac{d\gamma}{dt},\ \frac{d\gamma\bar{\beta}}{dt}\right)$$

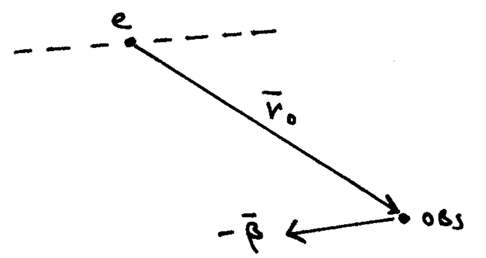
IN PARTICULAR, $m_0\frac{d\gamma\bar{v}}{dt} = \frac{d\bar{p}}{dt} = e\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right)$, AND $m_0c^2\frac{d\gamma}{dt} = \frac{dU}{dt} = e\bar{v}\cdot\bar{E}$

HOLDS IN ANY FRAME.

THE POTENTIAL OF A MOVING CHARGE



LAB FRAME



INSTANTANEOUS REST FRAME OF THE PARTICLE.

IN THE PARTICLE'S REST FRAME, $\phi_\mu = \left(\frac{e}{r_0},\ \bar{0}\right)$

IN THE LAB FRAME WE CAN QUICKLY TRANSFORM THIS TO

$$\phi_\mu = \left(\frac{\gamma e}{r_0},\ \frac{\gamma e\bar{\beta}}{r_0}\right)$$

THIS IS GREAT, EXCEPT THAT r_0 IS MEASURED IN THE REST FRAME, NOT THE LAB FRAME.

WE FIRST NOTE THAT IT PROBABLY MAKES MOST SENSE TO TRY TO DEFINE THE LABORATORY POTENTIALS IN TERMS OF THE RETARDED POSITION OF THE CHARGE.

IN THE LAB, THE RETARDED TIME IS THEN $t_{obs} - r/c$ OR

$$\Delta t \equiv t_{ret} - t_{obs} = -r/c$$

THEN REPPRESENTING THE RETARDED POSITION OF THE CHARGE AS A 4-VECTOR

$$\Delta x_\mu = (c\Delta t, -\bar{r}) = (-r, -\bar{r}) \qquad \text{(SIGN OF } \bar{r} \text{ AS IN FIGURE)}$$

WHICH IS A LIGHT-LIKE 4-VECTOR, OF COURSE.

HENCE, IN THE REST FRAME THIS 4-VECTOR MUST HAVE COMPONENTS

$$\Delta x_\mu' = (-r_0, -\bar{r}_0) \qquad \text{SO AS TO BE LIGHT-LIKE STILL.}$$

IT IS EASIER TO USE THE LORENTZ TRANSFORMATION FOR THE TIME COMPONENT OF THIS 4-VECTOR TO RELATE r TO r_0 ...

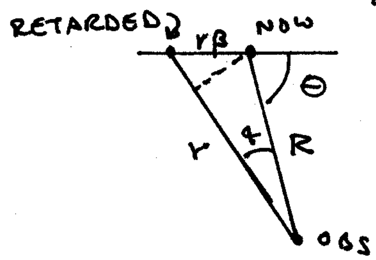
$$\Delta x_0' = \gamma(\Delta x_0 - \bar{\beta}\cdot\Delta\bar{x}) \Rightarrow -r_0 = \gamma(-r + \bar{\beta}\cdot\bar{r})$$

HENCE $$\phi_\mu = \left(\left[\frac{e}{r - \bar{\beta}\cdot\bar{r}}\right], \left[\frac{e\bar{\beta}}{r - \bar{r}\cdot\bar{\beta}}\right]\right) \quad \text{IN THE LAB}$$

[COMPARE BECKER 66.9

$[r]$ = RETARDED POSITION, ETC.

EXAMPLE: IF THE PARTICLE'S VELOCITY IS CONSTANT, WE MAY PREFER TO EXPRESS ϕ_μ IN TERMS OF THE PRESENT POSITION OF THE PARTICLE [BECKER SEC. 64



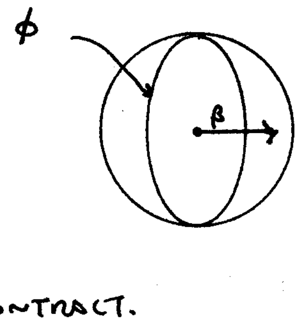
DURING THE RETARDED TIME r/c THE PARTICLE MOVES BY $\frac{r}{c}v = r\beta$ AS SHOWN

THEN $r - \bar{\beta}\cdot\bar{r} = R\cos\alpha = R\sqrt{1-\beta^2\sin^2\theta}$

AND $\dfrac{\sin\alpha}{r\beta} = \dfrac{\sin(\pi-\theta)}{r}$

$$\text{SO} \quad \phi_\mu = \left(\frac{e}{R\sqrt{1-\beta^2\sin^2\theta}}, \frac{e\bar{\beta}}{R\sqrt{\quad}}\right)$$

THE EQUIPOTENTIAL SURFACE IS THUS AN ELLIPSOID FLATTENED IN THE DIRECTION OF MOTION - BY THE LORENTZ CONTRACTION!



THIS WAS TO BE EXPECTED IF BODIES MADE OF ATOMS ARE TO SHOW THE LORENTZ CONTRACTION. THE SHAPE OF THE ATOM IS DEFINED BY SOME EQUIPOTENTIAL SURFACE - SO THAT SURFACE MUST CONTRACT.

IF $\bar{E} = -\bar{\nabla}\phi$ ONLY, THE FIELD LINES COULD NOT BE RADIAL! BUT $\frac{\partial \bar{A}}{\partial t} \neq 0$. ON THE PROBLEM SET YOU SHOULD SHOW THAT $\bar{E}$ IS IN FACT RADIAL.

Heaviside's Derivation of the Fields of a Uniformly Moving Charge

[Phil. Mag. 27, 324 (1889). This is an interesting example of the use of Galilean relativity, prior to Einstein. Here we follow a simpler version given by J.J. Thomson also in 1889.

The current of a charge density ρ that has velocity $\bar v$ is $\bar J = \rho \bar v$. Hence,

$\bar\nabla \times \bar B = \frac{4\pi}{c}\rho\bar v + \frac{1}{c}\frac{\partial \bar E}{\partial t}$ and $\bar\nabla \times \bar E = -\frac{1}{c}\frac{\partial \bar B}{\partial t}$ combine to give

$$\nabla^2 \bar B - \frac{1}{c^2}\frac{\partial^2 \bar B}{\partial t^2} = -\frac{4\pi}{c}\bar\nabla \times \rho \bar v$$ using $\bar\nabla \cdot \bar B = 0$.

This suggests use of the vector potential $\bar A$. With $\bar B = \bar\nabla \times \bar A$ we have

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 \bar A}{\partial t^2} = -\frac{4\pi}{c}\rho\bar v.$$

With $\bar v = v\hat z$ only A_z is non zero.

Following ~~Heaviside~~ Thomson, we try to convert this to an electrostatic problem by switching to a frame in which the charge is at rest. Not having heard of Einstein, we simply introduce $z' = z - vt$ and expect $A_z = A_z(x, y, z')$

Now $\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$ and $\frac{\partial}{\partial t} = -v\frac{\partial}{\partial z'}$, so the wave equation for A_z becomes

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 A_z}{\partial z'^2} = -\frac{4\pi}{c}\rho v.$$

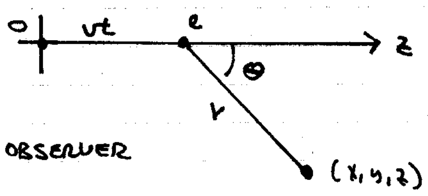
If we introduce $\tilde z = z'/\sqrt{1 - v^2/c^2}$ this will have the desired form:

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial \tilde z^2} = -\frac{4\pi}{c}\rho v.$$ Poisson's equation for charge $-\frac{\rho v}{c}$

The formal solution is then $A_z(x_0, y_0, \tilde z_0) = \int \frac{\rho v \, d\widetilde{vol}}{c \tilde r} = \frac{1}{c}\int \frac{\rho v \, dx\, dy\, d\tilde z}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z' - z_0')^2/(1 - v^2/c^2)}}$

For a point charge e at $(x, y, z') = (0,0,0)$, $\rho = e\delta^3(x, y, z')$ & we find

$$A_z(x_0, y_0, z_0') = \frac{ev/c}{\sqrt{(1 - v^2/c^2)(x_0^2 + y_0^2) + z_0'^2}}$$



Returning to (x, y, z) coords, we drop the subscript 0 for the observer and introduce $r = \sqrt{x^2 + y^2 + (z - vt)^2}$. Then

$$A_z(x, y, z, t) = \frac{ev/c}{r\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}}$$ where $\sin\theta = \frac{\sqrt{x^2 + y^2}}{r}$

$\bar B = \bar\nabla \times \bar A$. Switching to cylindrical coords (ρ, ϕ, z) we see that only B_ϕ is non zero

and $B_\phi = -\frac{\partial A_z}{\partial \rho} = e\frac{\frac{v}{c}\sin\theta\,(1 - v^2/c^2)}{r^2\,(1 - \frac{v^2}{c^2}\sin^2\theta)^{3/2}}$ noting that $\rho = \sqrt{x^2 + y^2}$

Heaviside then obtained $\bar E$ from $\bar B$ using arguments that are perhaps not fully justifiable.

a) $\bar{E}$ IS RADIAL WITH RESPECT TO ITS PRESENT POSITION. THE ARGUMENT IS PERHAPS THAT IN THE MOVING FRAME THE STATIC ELECTRIC FIELD IS RADIAL, SO THE TRANSFORMATION BACK TO THE ORIGINAL FRAME DOES NOT ALTER THIS.

b) MORE CONTROVERSIAL IS THE ARGUMENT FOR THE (TRUE) RELATION THAT

$\bar{B} = \bar{v}/c \times \bar{E}$. HEAVISIDE'S ARGUMENT APPEARS TO DEPEND ON THE EXISTENCE OF AN ETHER.

IF WE ACCEPT THESE TWO RESULTS, THEN $B_\phi = \frac{v}{c} E \sin\theta$ HOLDS FOR THE MAGNITUDES, LEADING TO

$$E \text{ (ALONG } \bar{r}) = \frac{e}{r^2} \frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2} \sin^2\theta\right)^{3/2}}.$$

(AS YOU SHOULD VERIFY ON THE PROBLEM SET BY OTHER METHODS.)

HEAVISIDE DID NOT USE THE SCALAR POTENTIAL IN HIS ARGUMENT, BUT DID NOTE THAT E AS FOUND IS NOT $-\frac{1}{c}\frac{\partial \bar{A}}{\partial t}$.

HE THEN CONSIDERED THE CASE OF A CHARGED SPHERICAL CONDUCTOR WITH VELOCITY $\bar{v}$. AFTER SOME DEBATE WITH G. SEARLE IT EMERGED (1890) THAT THE CHARGE DISTRIBUTION ON THE 'SPHERE' COULD REMAIN STATIC ONLY IF THE 'SPHERE' WERE REALLY AN OBLATE SPHEROID, SHORTER IN THE DIRECTION OF MOTION BY A FACTOR $\sqrt{1 - v^2/c^2}$.

THIS SEEMS TO BE THE FIRST RECORDED RESULT OF A 'LORENTZ' CONTRACTION FOR A PHYSICAL SYSTEM.

HOWEVER, HEAVISIDE NEVER APPRECIATED THE DEEP SIGNIFICANCE OF THIS RESULT, AND NEVER ABANDONED HIS BELIEF IN THE ETHER — ALTHOUGH HE WENT ON TO BE THE FIRST TO CALCULATE THE RADIATION FROM ACCELERATED CHARGES AS $v \to c$, AND ARGUED FOR THE EXISTENCE OF WHAT WE NOW CALL ČERENKOV RADIATION, THE TOPICS OF LECTURES 19 & 20.

HEAVISIDE'S FRIEND, FITZGERALD WENT ON TO ARGUE THAT THE CONTRACTION FACTOR WAS A UNIVERSAL PHENOMENON, AND HELPED UNDERSTAND THE NULL RESULT OF THE MICHELSON-MOREY EXPT. (1892)

[SEE PANOFSKY & PHILLIPS, 2ND ED, SEC 19.3, OR BECKER, SEC 64, FOR MODERNIZED VERSIONS OF HEAVISIDE'S DERIVATION.]

It may be remarked that (if my calculations are correct) equation (7) or its equivalents expresses the mutual energy of any two rational current-elements (see § 1) in a medium of uniform inductivity, of moments q_1u_1 and q_2u_2, whether the currents be of displacement, or conduction, or convection, or all mixed, it being in fact the mutual energy of a pair of definite magnetic fields. But, since the hypothesis of instantaneous action is expressly involved in the above, the application of (7) is of a limited nature.

General Theory of Convection Currents.

8. Now leaving behind altogether the subject of current-elements, in the investigation of which one is liable to be led away from physical considerations and become involved in mere exercises in differential coefficients, and coming to the question of the electromagnetic effects of a charge moving in any way, I have been agreeably surprised to find that my solution in the case of steady rectilinear motion, originally an infinite series of corrections, easily reduces to a very simple and interesting finite form, provided u be not greater than v. Only when $u > v$ is there any difficulty. We must first settle upon what basis to work. First the Faraday-law (p standing for d/dt),

$$-\operatorname{curl}\mathbf{E} = \mu_0 p\mathbf{H}, \quad \text{.........................(11)}$$

requires no change when there is moving electrification. But the analogous law of Maxwell, which I understand to be really a *definition* of electric current in terms of magnetic force, (or a doctrine), requires modification if the true current is to be

$$\mathbf{C} + p\mathbf{D} + \rho\mathbf{u}; \quad \text{.............................(12)}$$

viz., the sum of conduction-current, displacement-current, and convection-current $\rho\mathbf{u}$, where ρ is the volume-density of electrification. The addition of the term $\rho\mathbf{u}$ was, I believe, proposed by G. F. Fitzgerald.*

(This was not meant exactly for a new proposal, being in fact after Rowland's experiments; besides which, Maxwell was well acquainted with the idea of a convection-current. But what is very strange is that Maxwell, who insisted so strongly upon his doctrine of the *quasi*-incompressibility of electricity, never formulated the convection-current in his treatise. Now Prof. Fitzgerald pointed out that if Maxwell, in his equation of mechanical force,

$$\mathbf{F} = \mathrm{V}\mathbf{CB} - e\nabla\Psi - m\nabla\Omega,$$

had written $\mathbf{E}$ for $-\nabla\Psi$, as it is obvious he should have done, then the inclusion of convection-current in the true current would have followed naturally. (Here $\mathbf{C}$ is the true current, $\mathbf{B}$ the induction, e the density of electrification, m that of imaginary magnetic matter, Ψ the electrostatic and Ω the magnetic potential, and $\mathbf{E}$ the real electric force.)

Now to this remark I have to add that it is as unjustifiable to derive $\mathbf{H}$ from Ω as $\mathbf{E}$ from Ψ; that is, in general, the magnetic force is not the slope of a scalar potential; so, for $-\nabla\Omega$ we should write $\mathbf{H}$, the real magnetic force.

* Brit. Assoc., Southport, 1883.

O. HEAVISIDE, ELECTRICAL PAPERS. VOL. II.

But this is not all. There is possibly a fourth term in $\mathbf{F}$, expressed by $4\pi\mathrm{V}\mathbf{DG}$, where $\mathbf{D}$ is the displacement and $\mathbf{G}$ the magnetic current; I have termed this force the "magneto-electric force," because it is the analogue of Maxwell's "electromagnetic force," $\mathrm{V}\mathbf{CB}$. Perhaps the simplest way of deriving it is from Maxwell's electric stress, which was the method I followed.*

Thus, in a homogeneous nonconducting dielectric free from electrification and magnetization, the mechanical force is the sum of the "electromagnetic" and the "magnetoelectric," and is given by

$$\mathbf{F} = \frac{1}{v^2}\frac{d\mathbf{W}}{dt},$$

where $\mathbf{W} = \mathrm{V}\mathbf{EH}/4\pi$ is the transfer-of-energy vector.

It must, however, be confessed that the real distribution of the stresses, and therefore of the forces, is open to question. And when ether is the medium, the mechanical force in it, as for instance in a light-wave, or in a wave sent along a telegraph-circuit, is not easily to be interpreted.)

The companion to (11) in a nonconducting dielectric is now

$$\operatorname{curl}\mathbf{H} = cp\mathbf{E} + 4\pi\rho\mathbf{u}. \quad \text{.........................(13)}$$

Eliminate $\mathbf{E}$ between (11) and (13), remembering that $\mathbf{H}$ is circuital, because μ_0 is constant, and we get

$$(p^2/v^2 - \nabla^2)\mathbf{H} = \operatorname{curl} 4\pi\rho\mathbf{u}, \quad \text{.......................(14)}$$

the characteristic of $\mathbf{H}$. Here $\nabla^2 = d^2/dx^2 + \dots$, as usual.

Comparing (14) with the characteristic of $\mathbf{H}$ when there is impressed force $\mathbf{e}$ instead of electrification ρ, which is

$$(p^2/v^2 - \nabla^2)\mathbf{H} = \operatorname{curl} cp\mathbf{e},$$

we see that $\rho\mathbf{u}$ becomes $cp\mathbf{e}/4\pi$. We may therefore regard convection-current as *impressed* electric current. From this comparison also, we may see that an infinite plane sheet of electrification of uniform density cannot produce magnetic force by motion perpendicular to its plane. Also, we see that the sources of disturbances when ρ is moved are the places where $\rho\mathbf{u}$ has curl; for example, a dielectric sphere uniformly filled with electrification (which is imaginable), when moved, starts the magnetic force solely upon its boundary.

The presence of "curl" on the right side tells us, as a matter of mathematical simplicity, to make $\mathbf{H}$/curl the variable. Let

$$\mathbf{H} = \operatorname{curl}\mathbf{A}, \quad \text{...............................(15)}$$

and calculate $\mathbf{A}$, which may be any vector satisfying (15). Its characteristic is

$$(p^2/v^2 - \nabla^2)\mathbf{A} = 4\pi\rho\mathbf{u}. \quad \text{.........................(16)}$$

The divergence of $\mathbf{A}$ is of no moment, and it is only vexatious complication to introduce Ψ. The time-rate of decrease of $\mathbf{A}$ is not the real

* "El. Mag. Ind. and its Prop." XXII. *The Electrician*, Jan. 15, 1886, p. 187 [vol. I., p. 545].

distribution of electric force, which has to be found by the additional datum

$$\operatorname{div} c\mathbf{E} = 4\pi\rho, \qquad (17)$$

where $\mathbf{E}$ is the real force.

9. "Symbolically" expressed, the solution of (16) is

$$\mathbf{A} = \frac{4\pi\rho\mathbf{u}}{p^2/v^2 - \nabla^2} = \frac{-4\pi\rho\mathbf{u}/\nabla^2}{1 - p^2/v^2\nabla^2}. \qquad (18)$$

Here the numerator of the fraction to the right is the vector-potential of the convection-current. Calling it $\mathbf{A}_0$, we have

$$\mathbf{A}_0 = \frac{4\pi\rho\mathbf{u}}{-\nabla^2} = \sum \frac{\rho\mathbf{u}}{r}. \qquad (19)$$

Inserting in (18) and expanding, we have

$$\mathbf{A} = \left\{1 + (p/v\nabla)^2 + (p/v\nabla)^4 + \dots\right\}\mathbf{A}_0. \qquad (20)$$

Given then $\rho\mathbf{u}$ as a function of position and time, $\mathbf{A}_0$ is known by (19), and (20) finds $\mathbf{A}$, whilst (15) finds $\mathbf{H}$.

Complete Solution in the Case of Steady Rectilinear Motion. Physical Inanity of Ψ.

10. When the motion of the electrification is all in one direction, say parallel to the z-axis, $\mathbf{u}$, $\mathbf{A}_0$, and $\mathbf{A}$ are all parallel to this axis, so that we need only consider their tensors. When there is simply one charge q at a point, we have

$$A_0 = qu/r,$$

and (20) becomes

$$A = q\left\{1 + (p/v\nabla)^2 + (p/v\nabla)^4 + \dots\right\}(u/r) \qquad (21)$$

at distance r from q. When the motion is steady, and the whole electromagnetic field is ultimately steady with respect to the moving charge, we shall have, taking it as origin,

$$p = -u(d/dz) = -uD,$$

for brevity; so that

$$A = qu\left\{1 + (uD/v\nabla)^2 + (uD/v\nabla)^4 + \dots\right\}r^{-1}. \qquad (22)$$

Now the property $\quad \nabla^2 r^{n+2} = (n+2)(n+3)r^n \qquad (23)$

brings (22) to $\quad A = qu\left\{\frac{1}{r} + \frac{u^2}{v^2}D^2\frac{r}{2!} + \frac{u^4}{v^4}D^4\frac{r^3}{4!} + \dots\right\}; \qquad (24)$

and the property $\quad D^{2n}r^{2n-1} = 1^2.3^2.5^2\dots(2n-1)^2\nu^{2n}/r, \qquad (25)$

where $\nu = \sin\theta$, θ being the angle between $\mathbf{r}$ and the axis, brings (24) to

$$A = \frac{qu}{r}\left\{1 + \frac{u^2}{v^2}\,\frac{\nu^2}{2}\left(1 + \frac{u^2}{v^2}\,\frac{3}{4}\nu^2\left(1 + \frac{u^2}{v^2}\,\frac{5}{6}\nu^2\left(1 + \dots\right\}\right.\right.\right.; \qquad (26)$$

which, by the Binomial Theorem, is the same as

$$A = (qu/r)\{1 - u^2\nu^2/v^2\}^{-\frac{1}{2}}, \qquad (27)$$

the required solution.

11. To derive H, the tensor of the circular $\mathbf{H}$, let $r\nu = h$, the distance from the axis. Then, by (15),

$$H = -\frac{dA}{dh} = -\nu\frac{dA}{dr} + \frac{\mu\nu}{r}\frac{dA}{d\mu} = \frac{qu\nu}{r^2}\left(1 + \mu\frac{d}{d\mu}\right)\left(1 - \frac{u^2}{v^2}\nu^2\right)^{-\frac{1}{2}}, \qquad (28)$$

by (27), if $\mu = \cos\theta$. Performing the differentiation, and also getting out E, the tensor of the electric force, we have the final result that the electromagnetic field is fully given by *

$$cE = \frac{q}{r^2}\cdot\frac{1 - u^2/v^2}{(1 - u^2\nu^2/v^2)^{\frac{3}{2}}}, \qquad H = cEu\nu, \qquad (29)$$

with the additional information that $\mathbf{E}$ is radial and $\mathbf{H}$ circular.

Now, as regards Ψ, if we bring it in, we have only got to take it out again. When the speed is very slow we may regard the electric field as given by $-\nabla\Psi$ *plus* a small correcting vector, which we may call the electric force of inertia. But to show the *physical* inanity of Ψ, go to the other extreme, and let u nearly equal v. It is now the electric force of inertia (supposed) that equals $+\nabla\Psi$ nearly (except about the equatorial plane), and its sole utility or function is to cancel the other $-\nabla\Psi$ of the (supposed) electrostatic field. It is surely impossible to attach any physical meaning to Ψ and to propagate it, for we require two Ψ's, one to cancel the other, and both propagated infinitely rapidly.

As the speed increases, the electromagnetic field concentrates itself more and more about the equatorial plane, $\theta = \frac{1}{2}\pi$. To give an idea of the accumulation, let $u^2/v^2 = \cdot 99$. Then cE is $\cdot 01$ of the normal value q/r^2 at the pole, and 10 times the normal value at the equator. The latitude where the value is normal is given by

$$\nu = (v/u)\left[1 - (1 - u^2/v^2)^{\frac{2}{3}}\right]^{\frac{1}{2}}. \qquad (30)$$

Limiting Case of Motion at the Speed of Light. Application to a Telegraph Circuit.

12. When $u = v$, the solution (29) becomes a plane electromagnetic wave, E and H being zero everywhere except in the equatorial plane. As, however, the values of E and H are infinite, distribute the charge along a straight line moving in its own line, and let the linear-density be q. The solution is then †

$$H = Ecv = 2qv/r \qquad (31)$$

at distance r from the line, between the two planes through the ends of the line perpendicular to it, and zero elsewhere.

To further realize, let the field terminate internally at $r = a$, giving a cylindrical-surface distribution of electrification, and terminate the tubes

* *The Electrician*, Dec. 7, 1888, p. 148 [p. 495, vol. II.].

† Ibid., Nov. 23, 1888, p. 84 [p. 493, vol. II.].