Electromagnetic Mass & The Radiation Reaction (Becker sec's 65, 83, 91)

One of the themes of this course has been the development of Maxwell's equations for point charges as a means of gaining a deeper understanding of the electric properties of macroscopic matter. (Lorentz is the major figure in the historical development of this approach.) We now want to examine some difficulties with this approach, concerning the energy of point particles. This is one area in which we cannot appeal to quantum mechanics for the eventual resolution of the difficulties, for they reappear there as well!

We have already remarked on the blow-up of the electrostatic field energy of a point charge. Considering the point charge to be the limit as $a \to 0$ of a spherical shell of charge of radius a.

$$U = \frac{1}{8\pi}\int E^2 \, d\text{vol} = \frac{e^2}{2}\int_a^\infty \frac{dr}{r^2} = \frac{e^2}{2a} \to \infty \quad \text{as } a \to 0$$

Following Einstein, we are tempted to identify a mass with this energy, according to

$$M_{\text{effective}} = \frac{U}{c^2} = \frac{1}{2}\frac{e^2}{c^2 a}$$

We have previously remarked how the quantity

$$r_0 = \frac{e^2}{Mc^2} \sim 2.8 \times 10^{-13}\,\text{cm}$$

might have the significance of being the "classical electron radius."

But even prior to Einstein it was recognized that the field energy might have such a relation to the particle mass. Already in 1881 J.J. Thomson gave a non-relativistic demonstration - of which we now give the relativistic version.

We wish to evaluate the field energy and momentum of a charge moving with uniform velocity $\bar{v}$. When at rest, the charge is a spherical shell of radius a.

$$\text{Energy} \qquad U = \frac{1}{8\pi}\int (E^2 + B^2)\, d\text{vol}$$

$$\text{Momentum} \qquad \bar{P} = \int \frac{\bar{S}}{c^2}\, d\text{vol} = \frac{1}{4\pi c}\int \bar{E}\times\bar{B}\, d\text{vol}$$

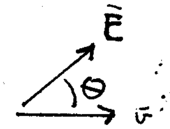
These integrals are complicated by the fact that the volume of integration is everywhere outside the ellipsoid which results from the Lorentz contraction of a sphere. Furthermore, the expressions for $\bar{E}$ and $\bar{B}$ of a moving charge are somewhat cumbersome.

WE CAN PLAY A TRICK: CONVERT THE INTEGRAND TO QUANTITIES IN THE PARTICLE'S REST FRAME!

IF $\bar{v}$ IS ALONG THE Z-AXIS, THEN $dx = dx^*$, $dy = dy^*$

BUT $dz = \frac{dz^*}{\gamma}$ BUT TO THE LORENTZ CONTRACTION

THUS $d\text{vol} = dx\,dy\,dz = \frac{1}{\gamma}\, d\text{vol}^*$



AND $\int d\text{vol}^*$ IS AN INTEGRAL OVER THE VOLUME OUTSIDE A SPHERE!

ALSO $\bar{E}_\parallel = \bar{E}_\parallel^*$ $\quad \bar{E}_\perp = \gamma \bar{E}_\perp^*$ AND $\bar{B} = \gamma \frac{\bar{v}}{c} \times \bar{E}_\perp^*$

THUS $$\bar{p} = \frac{1}{4\pi c}\gamma^2 \frac{\bar{v}}{c} \int (E_\perp^*)^2 \frac{d\text{vol}^*}{\gamma} = \frac{\gamma \bar{v}}{4\pi c^2} e^2 \underbrace{\int_a^\infty \frac{2\pi r^2 dr}{r^4}}_{\frac{2\pi}{a}} \underbrace{\int_{-1}^{1} d\cos\theta \sin^2\theta}_{4/3}$$

OR $$\bar{p} = \frac{2}{3}\frac{e^2}{a c^2}\gamma \bar{v} = \underline{\frac{4}{3}\frac{U^*}{c^2}\gamma \bar{v}}$$

WHERE $U^* = \frac{e^2}{2a}$ AS BEFORE

THIS RESULT SUGGESTS THAT WE CAN IDENTIFY $\frac{4}{3}\frac{U^*}{c^2}$ AS THE EFFECTIVE MASS OF THE FIELD.

BUT WHAT ABOUT THE ENERGY U OF THE MOVING CHARGE?

$$U = \frac{1}{8\pi}\int \left[(E_\parallel^*)^2 + \gamma^2 (E_\perp^*)^2 + \gamma^2\beta^2 (E_\perp^*)^2 \right] \frac{d\text{vol}^*}{\gamma}$$ IN THE SAME WAY.

$$= \frac{1}{8\pi\gamma} 2\pi e^2 \int_a^\infty \frac{r^2 dr}{r^4} \int_{-1}^{1} d\cos\theta \left(\cos^2\theta + \gamma^2(1+\beta^2)\sin^2\theta \right)$$

$$= \frac{e^2}{4a\gamma}\left(\frac{2}{3} + \frac{4}{3}\gamma^2(1+\beta^2) \right) \qquad [\gamma^2\beta^2 = \gamma^2 - 1]$$

$$\underline{U = \frac{4}{3}U^*\gamma - \frac{U^*}{3\gamma}}$$

THIS WOULD BE REALLY SPLENDID WERE IT NOT FOR THE EXTRA PIECE $-\frac{U^*}{3\gamma}$. AS SUCH, $(U, \bar{p}c)$ DOES <u>NOT</u> SEEM TO BE A 4-VECTOR.

IN THE LIMIT $v/c \ll 1$, $\quad U \to U^*\left(\frac{4}{3}\left(1+\frac{\beta^2}{2}\right) - \frac{1}{3}\left(1-\frac{\beta^2}{2}\right)\right) = U^*\left(1 + \frac{5}{3c^2}\frac{v^2}{2}\right)$

SO $\text{K.E.} \doteq U - U^* = \frac{1}{2}\frac{5U^*}{3c^2}v^2$, WHICH SUGGESTED TO THOMSON THAT $m \doteq \frac{5}{3}U^*/c^2$

THESE ARGUMENTS GIVE A SENSE OF PLAUSIBILITY TO THE IDENTIFICATION

$$m \sim \frac{U^*}{c^2}$$

BUT DO NOT SEEM TO BE ABSOLUTELY CONVINCING.

THE IDEA THAT PART OR ALL OF THE MASS OF A CHARGED PARTICLE MIGHT BE DUE TO ITS ELECTROMAGNETIC FIELD IS SO ENTICING THAT PEOPLE HAVE TRIED HARD TO DEMONSTRATE ITS VALIDITY.

WE HAVE SHOWN THAT U AND $\bar{P}$ DO NOT TRANSFORM EXACTLY LIKE THE COMPONENTS OF A 4-VECTOR. THIS HAS LED ONE GROUP OF PEOPLE TO REDEFINE U AND $\bar{P}$ SO THAT THEY DO TRANSFORM PROPERLY. THIS MEANS ABANDONING THE DERIVATION OF U AND $\bar{P}$ DUE TO POYNTING — WHICH IS A PRE-RELATIVITY ARGUMENT. WE MUST JUST SAY THAT IF A CHARGED PARTICLE HAS VELOCITY $\bar{v}$, THEN

$$U = \gamma U^* \qquad \text{AND} \qquad \bar{P} = \gamma U^* \bar{v} \qquad \text{BY DEFINITION.}$$

TO ME THIS IS UNSATISFACTORY BECAUSE U AND $\bar{P}$ ARE NOT DEFINED DIRECTLY IN TERMS OF QUANTITIES MEASURED IN THE OBSERVER'S FRAME. INSTEAD WE MUST ALWAYS REFER TO THE REST FRAME OF THE PARTICLE WHEN DISCUSSING LAB FRAME MOMENTUM AND ENERGY. BUT IT IS A CONSISTENT PROCEDURE.

IN RECENT TIMES THIS VIEW IS ADVOCATED MOST STRONGLY BY ROHRLICH IN AN INTERESTING BOOK "CLASSICAL CHARGED PARTICLES." THE ARGUMENT IS SECONDED IN THE TEXT BY JACKSON.

A DIFFERENT VIEW MAY BE THE MORE ACCEPTED ONE. U AND $\bar{P}$ DO NOT TRANSFORM LIKE PIECES OF A 4-VECTOR BECAUSE THEY ARE ONLY PART OF THE TOTAL ENERGY AND MOMENTUM OF A PARTICLE.

$$U_{TOT} = U_{E\&M} + U_{OTHER}$$

$$\bar{P}_{TOT} = \bar{P}_{E\&M} + \bar{P}_{OTHER}$$

$$\text{SO THAT} \quad (M_{TOT}\, c^2)^2 = U_{TOT}^2 - P_{TOT}^2 c^2$$

BY A SLEIGHT OF HAND THIS ALLOWS US TO ACCOMODATE THE INFINITE ENERGY $U^* = \frac{e^2}{2a}$ AS $a \to 0$. SINCE $U_{TOT} = U_{E\&M} + U_{OTHER}$ WHY CAN'T BOTH $U_{E\&M}$ AND U_{OTHER} BE INFINITE IN SUCH A WAY THAT U_{TOT} (OR M_{TOTAL}) IS FINITE? THIS IS THE IDEA OF <u>MASS RENORMALIZATION</u>, WHICH HAS PROVED EFFECTIVE IN QUANTUM ELECTRODYNAMICS, BUT CAN BE APPLIED IN CLASSICAL SITUATIONS AS WELL. (KRAMERS, 1947)
[J. SCHWINGER, FOUND. PHYS. 13, 373 (1983).]

Poincaré Stresses

The concept of mass renormalization is a descendent of a suggestion made by Poincaré in 1906. He remarked that a spherical shell of charge cannot be held together by electrical forces alone. There must be some additional attractive force. This force should also be related to an energy density which may indeed make $(U_{TOT}, \bar{P}_{TOT}c)$ into a 4-vector.

To sketch Poincaré's argument we must introduce the 4-tensor version of the Maxwell stress tensor discussed in Lecture 10 (see Becker sec 83). Our method is a direct relativistic generalisation of that used previously.

In Lecture 18 we saw how the Lorentz force on a current distribution can be written

$$f_\mu = F_{\mu\nu} j^\nu = \left(\frac{\bar{J}\cdot\bar{E}}{c}, \rho\bar{E} + \frac{\bar{J}}{c}\times\bar{B}\right)$$

The time component of the force density 4-vector is the power density.

The goal is to replace volume integrals of f_μ by surface integrals of the stress-tensor $T_{\mu\nu}$. As in Lecture 3, this can be possible, according to Gauss' theorem, if $f_\mu = -\dfrac{\partial T_{\mu\nu}}{\partial x_\nu} = -\partial^\nu T_{\mu\nu}$

where we introduce a minus sign to beautify things later. $\left[\partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, -\bar{\nabla}\right)\right]$

Now 2 of Maxwell's equations can be written

$$j_\nu = \frac{1}{4\pi}\partial^\mu F_{\mu\nu}$$

$$\text{so } f_\mu = F_{\mu\nu} j^\nu = \frac{1}{4\pi} F_{\mu\nu}\partial^\alpha F_\alpha{}^\nu = \frac{1}{4\pi}\partial^\alpha\left(F_{\mu\nu}F_\alpha{}^\nu\right) - \frac{1}{4\pi}F_\alpha{}^\nu\partial^\alpha F_{\mu\nu}$$

$$\text{Further, } F_\alpha{}^\nu\partial^\alpha F_{\mu\nu} = \frac{1}{2}F_\alpha{}^\nu\partial^\alpha F_{\mu\nu} + \frac{1}{2}F_\nu{}^\alpha\partial^\nu F_{\mu\alpha}$$

↖ swap α & ν

$$= \frac{1}{2}F_\alpha{}^\nu\partial^\alpha F_{\mu\nu} + \frac{1}{2}\left(-F^\alpha{}_\nu\right)\partial^\nu\left(-F_{\alpha\mu}\right)$$

↖ since $F_{\mu\nu} = -F_{\nu\mu}$

$$= \frac{1}{2}F_\alpha{}^\nu\left(\partial^\alpha F_{\mu\nu} + \partial_\nu F^\alpha{}_\mu\right)$$

Now the other two Maxwell equations are $\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0$

$$\text{so } F_\alpha{}^\nu\partial^\alpha F_{\mu\nu} = -\frac{1}{2}F_\alpha{}^\nu\partial_\mu F_\nu{}^\alpha = -\frac{1}{4}\partial_\mu\left(F_\alpha{}^\nu F_\nu{}^\alpha\right)$$

WE MAY WRITE $\partial_\mu = \partial_\beta \eta_\mu{}^\beta$ WHERE $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

THEN ALTOGETHER, $f_\mu = \frac{1}{4\pi} \partial^\alpha \left(F_{\mu\nu} F_\alpha{}^\nu\right) + \frac{1}{16\pi} \partial^\beta \eta_{\mu\beta} \left(F_\alpha{}^\nu F_\nu{}^\alpha\right)$

$$= \partial^\nu \left(\frac{1}{4\pi} F_{\mu\alpha} F_\nu{}^\alpha + \frac{1}{16\pi} \eta_{\mu\nu} F_\alpha{}^\beta F_\beta{}^\alpha \right)$$

$$\equiv - \partial^\nu T_{\mu\nu}$$

SO $\underline{T_{\mu\nu} = \frac{1}{4\pi} F_{\mu\alpha} F^\alpha{}_\nu + \frac{1}{16\pi} \eta_{\mu\nu} F_\alpha{}^\beta F_\alpha{}^\beta}$

THIS IS SYMMETRIC IN μ AND ν. (WE USED THE FACT THAT $F_{\alpha\beta}$ IS ANTISYMMETRIC)

IN LECTURE 18 WE SAW THAT $F_\alpha{}^\beta F_\alpha{}^\beta = F_{\alpha\beta} F^{\alpha\beta} = 2(B^2 - E^2)$

WORKING OUT THE COMPONENTS OF $T_{\mu\nu}$ IN DETAIL, WE FIND

$$T_{\mu\nu} = \left(\begin{array}{c|c} \frac{E^2+B^2}{8\pi} & \frac{1}{4\pi} \bar{E} \times \bar{B} = \frac{\bar{S}}{c} \\ \hline \frac{1}{4\pi} \bar{E} \times \bar{B} & -\frac{1}{4\pi} \underbrace{\left[E_i E_j + B_i B_j - \delta_{ij} \frac{E^2+B^2}{2} \right]}_{-\overline{\overline{T}}} \end{array} \right)$$

WHICH NICELY COMBINES THE ENERGY DENSITY, THE POYNTING VECTOR AND THE 3 x 3 MAXWELL STRESS TENSOR INTO A SINGLE ENTITY!

WE CAN PLAY WITH THIS A BIT AND RECOVER THE CONSIDERATIONS OF LECTURE 10.

FOR EXAMPLE, SUPPOSE WE INTEGRATE f_0 OVER A VOLUME

$$\int f_0 \, d\text{vol} = \frac{1}{c} \int \bar{J} \cdot \bar{E} \, d\text{vol} = - \int \partial^\nu T_{0\nu} \, d\text{vol} = -\frac{1}{c} \frac{\partial}{\partial t} \int u \, d\text{vol} - \frac{1}{c} \int \bar{\nabla} \cdot \bar{S} \, d\text{vol}$$

WITH $u = \frac{E^2+B^2}{8\pi}$ AND $\bar{S} = \frac{c}{4\pi} E \times \bar{B}$

$$= -\frac{1}{c} \frac{\partial}{\partial t} \int u \, d\text{vol} - \frac{1}{c} \int \bar{S} \cdot d\overline{\text{AREA}}$$

IN OTHER WORDS: POWER CONSUMED = RATE OF LOSS OF STORED ENERGY + FLOW OF ENERGY FROM THE OUTSIDE.

SIMILARLY

$$\bar{F}_{MECH}\Big|_i = \int \bar{f}\, dvol\Big|_i = \int \left(\rho \bar{E} + \frac{1}{c}\bar{J}\times\bar{B}\right) dvol\Big|_i = -\int \partial^\nu T_{i\nu}\, dvol = -\frac{1}{4\pi c}\frac{\partial}{\partial t}\int \bar{E}\times\bar{B}\Big|_i dvol + \int \bar{\nabla}\cdot\bar{\bar{T}}\, dvol\Big|_i$$

$$\bar{F}_{MECH} = \frac{d\bar{P}_{MECH}}{dt} = -\frac{\partial}{\partial t}\bar{P}_{FIELD} + \int \bar{\bar{T}}\cdot d\bar{A}_{REA} \qquad \text{WITH } \bar{P}_{FIELD} = \frac{1}{4\pi c}\int \bar{E}\times\bar{B}\, dvol$$

AND $\bar{\bar{T}}$ = 3x3 MAXWELL-STRESS TENSOR.

HENCE: RATE OF CHANGE OF TOTAL MOMENTUM = INTEGRAL OF 3-STRESS TENSOR OVER SURFACE

THUS THE ORIGINAL DERIVATION OF THE POYNTING VECTOR HAS BEEN RECAST INTO RELATIVISTIC FORM.

WE NOW WISH TO RECONSIDER THE QUESTION OF CONSTRUCTING AN <u>ENERGY</u>-<u>MOMENTUM</u> <u>4-VECTOR</u> FOR THE FIELDS.

WE FIRST NOTE THIS CAN BE DONE IF THE FIELDS ARE NOT TIED TO CHARGES, AS IS THE CASE FOR THE RADIATION FIELDS (ONCE THEY HAVE LEFT THE SOURCE).

WE DEFINE $P_\mu = \int T_{0\mu}\, dvol$

SO $P_0 = \int T_{00}\, dvol = \int u\, dvol$

$P_i = \int T_{0i}\, dvol = \frac{1}{4\pi}\int \bar{E}\times\bar{B}\, dvol = c\int \bar{P}_{FIELD}\, dvol$

SO $P_\mu = (U, c\bar{P})$ HAS THE APPEARANCE OF COMPONENTS OF A 4-VECTOR.

BUT IS IT REALLY A 4-VECTOR? THIS IS WHERE WE RAN INTO TROUBLE EARLIER.

<u>IF</u> THERE ARE NO SOURCES FOR THE FIELDS, (i.e., IN A CHARGE-FREE REGION) THEN THE LORENTZ FORCE

$f_\mu = 0$ CERTAINLY $\Rightarrow$ $\partial^\nu T_{\mu\nu} = 0$ ALSO.

IN THIS CASE P_μ AS DEFINED ABOVE IS ACTUALLY A 4-VECTOR.

A DEMONSTRATION OF THIS INVOLVES RECOGNIZING THAT

$$\int dvol\Big|_{\text{FIXED TIME}} = \int \text{4-DIMENSIONAL SURFACE},\ t = \text{CONST}$$

ALSO WE NEED GAUSS' THEOREM IN 4-DIMENSIONS

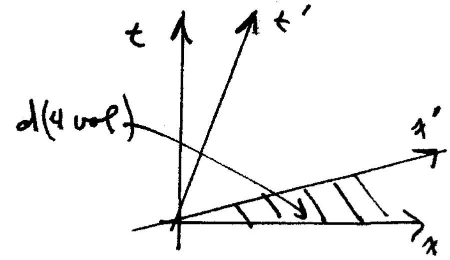
$$\int \partial^\mu A_\mu \, d(4\text{-vol}) = \int A^\mu \cdot d(4\text{-surface})_\mu$$

$d(4\text{-vol}) = c\,dt\,dx\,dy\,dz$

$d(4\text{-surface})_\mu = d3\text{-vol} \cdot$ (UNIT 4-VECTOR WHICH IS NORMAL TO THE SURFACE)

WE ARE MAINLY INTERESTED IN SURFACES OF FIXED TIME IN SOME INTERTIAL FRAME. IN THAT FRAME, THE UNIT NORMAL IS $(1,0,0,0)$ AND $d\,3\text{vol} = dx\,dy\,dz$

IF WE CONSIDER A 4-VOL WHICH IS BOUNDED BY SURFACES OF FIXED TIME IN TWO DIFFERENT FRAMES



THEN $\int A^\mu \cdot d(4\cdot\text{surface})_\mu$

$$= \int A_0 \, d\text{vol} - \int A_0' \, d\text{vol}'$$

+ TERMS AT ∞

WHERE A_0' = TIME COMPONENT OF A_μ IN FRAME $(\bar{x}', t')$ ETC.

SO IF $\partial^\mu A_\mu = 0$ THEN $\int A_0 \, d\text{vol} = \int A' d\text{vol}'$

LET $A_\mu = B^\nu T_{\mu\nu}$ WITH B^ν = CONSTANT 4-VECTOR

$$\partial^\mu A_\mu = B^\nu \partial^\mu T_{\mu\nu} = 0$$

SO $B^\nu \int T_{0\nu} \, d\text{vol} = B'^\nu \int T_{0\nu}' \, d\text{vol}'$ FOR ANY FRAMES AND FOR ANY B^ν

$\therefore$ $\int T_{0\nu} \, d\text{vol}$ MUST BE A 4 VECTOR!

HENCE THE FIELD 4-VECTOR $P_\mu = (U, \bar{P}c) = \int T_{0\mu} \, d\text{vol}$

HAS A WELL DEFINED RELATIVISTIC MEANING FOR WAVES IN FREE SPACE IN WHICH NO SOURCES ARE PRESENT.

BUT IF CHARGES ARE PRESENT, THIS ARGUMENT DOES NOT WORK.

WE CAN RECAST THE ARGUMENT GIVEN AT THE BEGINNING OF THIS LECTURE IN OUR NEW NOTATION:

$$U = \int T_{00} \, d\text{vol} \qquad \bar{P}_i = \frac{1}{c} \int T_{0i} \, d\text{vol}$$

SUPPOSE WE APPLY THIS TO A CHARGE, OR SYSTEM OF CHARGES. IN THE APPROPRIATE REST FRAME WE HAVE

$$U^* = \int T^*_{00}\, d\mathrm{vol}^* \qquad P_i^* = \frac{1}{c}\int T^*_{0i}\, d\mathrm{vol}^* = 0$$

IN SOME OTHER FRAME, MOVING WITH VELOCITY $\bar{v} = v\hat{x}$ WITH RESPECT TO THE REST FRAME, THE STRESS TENSOR HAS COMPONENTS

$$T_{\mu\nu} = L_{\mu\alpha} L_{\nu\beta} T^*_{\alpha\beta} \qquad \left[L_{\mu\nu} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right.$$

IN PARTICULAR

$$\begin{aligned} T_{00} &= L_{00}L_{00}T^*_{00} \\ &\quad \left.\begin{aligned} &+ L_{01}L_{00}T^*_{10} \\ &+ L_{00}L_{01}T^*_{01} \end{aligned}\right\} = 0 \quad \text{SINCE } T^*_{01} = 0 \text{ ON INTEGRATION OVER } d\mathrm{vol}^* \\ &\quad + L_{01}L_{01}T^*_{11} \\ &= \gamma^2\left(T^*_{00} + \beta^2 T^*_{11}\right) \end{aligned}$$

$$\begin{aligned} T_{01} &= L_{00}L_{10}T^*_{00} \\ &\quad + L_{01}L_{11}T^*_{11} \\ &= \gamma^2\beta\left(T^*_{00} + T^*_{11}\right) \end{aligned}$$

AS BEFORE $d\mathrm{vol} = \frac{1}{\gamma} d\mathrm{vol}^*$

SO $$U = \int T_{00}\, d\mathrm{vol} = \gamma \int \left(T^*_{00} + \beta^2 T^*_{11}\right) d\mathrm{vol}^*$$

$$c\bar{P}_1 = \int T_{01}\, d\mathrm{vol} = \gamma\beta \int \left(T^*_{00} + T^*_{11}\right) d\mathrm{vol}^*$$

NOW WE CAN SEE THAT IT IS THE NON-VANISHING T^*_{11} TERM WHICH RUINS THE CHANCE FOR $(U, \bar{P}c)$ TO BE A 4-VECTOR.

POINCARÉ'S REMARK IS THAT IF THE CHARGE IS TO HOLD TOGETHER THERE MUST BE SOME ADDITIONAL FORCE. THIS FORCE CAN BE ASSOCIATED WITH A STRESS TENSOR AS WELL, SAY $P_{\mu\nu}$ FOR POINCARÉ

IN THE REST FRAME THE CHARGE MUST BE AT EQUILIBRIUM UNDER THE COMBINED INFLUENCE OF THE ELECTROMAGNETIC STRESS AND THE POINCARE STRESS.

HENCE $$\int_{\text{SURFACE}} \left(T^*_{ij} + P^*_{ij}\right) d\,\mathrm{Area}^*_i = 0$$ FOR ANY SURFACE. $[i, j = 1,2,3]$

SO IN FACT $T^*_{ij} + P^*_{ij} = 0 \Rightarrow \int \left(T^*_{11} + P^*_{11}\right) d\mathrm{vol}^* = 0$

(OR, NOTE THAT THE FORCE DENSITY $f_\mu = \partial^\nu\left(T_{\mu\nu} + P_{\mu\nu}\right) = 0$ NOW
$\Rightarrow \int \left(T_{0\mu} + P_{0\mu}\right) d\mathrm{vol}$ IS A 4-VECTOR BY OUR GENERAL THEOREM)

THUS IN A GENERAL FRAME

$$U = \gamma \int (T_{00}^* + P_{00}^*)\, d\mathrm{vol}^* \qquad c\bar{P} = \gamma\beta \int (T_{00}^* + P_{00}^*)\, d\mathrm{vol}^*$$

WHICH IS A PROPER 4-VECTOR TRANSFORMATION AT LAST.

THE MASS ASSOCIATED WITH THESE FIELDS IS $\quad Mc^2 = U^* = \int (T_{00}^* + P_{00}^*)\, d\mathrm{vol}^*$

BUT IT IS NOT PURELY ELECTROMAGNETIC!

THE RADIATION REACTION (SEE SEC 4, VOL II OF BECKER)

SO FAR WE HAVE CONSIDERED THE TROUBLES WITH THE CONCEPT OF A POINT CHARGE AT REST OR IN UNIFORM MOTION. NOW WE LOOK AT THE QUESTION OF AN ACCELERATING POINT CHARGE.

THE ELECTROSTATIC SELF ENERGY OF A "POINT" CHARGE IS DIVERGENT ($e^2/a \to \infty$ AS $a \to 0$), SO WE EXPLORE THE POSSIBILITY THAT A "POINT" CHARGE HAS A SMALL BUT FINITE RADIUS a.

WHEN SUCH A CHARGE IS ACCELERATING, THE ELECTROMAGNETIC INTERACTION AMONG VARIOUS PORTIONS OF THE CHARGE RESULTS IN A SELF FORCE, FIRST ANALYZED BY LORENTZ, AND IN GREATER DETAIL BY ABRAHAM (1895-1905).

SUCH ANALYSES REMAIN CONTROVERSIAL. WE PRESENT ONLY SIMPLIFIED ARGUMENTS THAT ILLUSTRATE THE MAIN FEATURES.

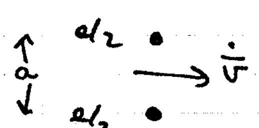
FOR A CHARGE THAT IS ACCELERATING, BUT INSTANTANEOUSLY AT REST, THE SELF FORCE IS

$$\bar{F}_{SELF} = \int \rho \bar{E}_{SELF}\, d\mathrm{vol}.$$

FOR A SYMMETRICAL CHARGE DISTRIBUTION, THE ELECTROSTATIC PART OF $\bar{E}$ DOES NOT CONTRIBUTE. BUT FROM THE VECTOR POTENTIAL,

$$\bar{E} = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t} = -\frac{1}{c^2}\int \frac{[\dot{\bar{J}}]}{r}\, d\mathrm{vol} \to -\frac{e\,[\dot{\bar{v}}]}{c^2 r}$$ FOR A SINGLE CHARGE e. (RECALL P. 178.)

A USEFUL MODEL OF CHARGE e IS A PAIR OF CHARGES $e/2$, SEPARATED BY DISTANCE a, $\perp$ TO $\dot{\bar{v}}$.



$$\text{THEN} \quad \bar{F}_{SELF} = 2\frac{e}{2}\left(-\frac{e\,[\dot{\bar{v}}]}{2c^2 a}\right) = -\frac{e^2}{2c^2 a}\dot{\bar{v}}\left(t - \frac{a}{c}\right) \approx \frac{-e^2}{2c^2 a}\left(\dot{\bar{v}} - \frac{a}{c}\ddot{\bar{v}}\right)$$

$$\approx -\frac{e^2\dot{\bar{v}}}{2c^2 a} + \frac{e^2\ddot{\bar{v}}}{2c^2}$$

THE EQUATION OF MOTION IS

$$m\dot{\bar{v}} = \bar{F}_{EXT} + \bar{F}_{SELF} \Rightarrow \left(m + \frac{e^2}{2c^2 a}\right)\dot{\bar{v}} = \bar{F}_{EXT} + \frac{e^2\ddot{\bar{v}}}{2c^2}.$$

Ph 206 LECTURE 22

THE DIVERGENT TERM, $\frac{e^2}{2c^2 a} \approx \frac{U_{ELECTROSTATIC}}{c^2} \equiv$ ELECTROMAGNETIC MASS, IS CONVENTIONALLY "RENORMALIZED" INTO THE TOTAL MASS.

THE SECOND NEW TERM, $\frac{e^2 \ddot{\bar{v}}}{2c^2}$, WAS INTERPRETED BY LORENTZ AS A KIND OF REACTION TO THE RADIATION OF THE ACCELERATED CHARGE.

IF AN EXTERNAL FORCE IS CAUSING THE CHARGE TO ACCELERATE, THIS FORCE MUST PROVIDE FOR THE RADIATED ENERGY OF THE FIELDS, AS WELL AS THE KINETIC ENERGY OF THE CHARGE. WE MAY ACCOMODATE THIS BY SUPPOSING, ACCORDING TO NEWTON'S 3RD LAW, THAT THE RADIATION EXERTS A REACTION FORCE BACK ON THE CHARGE.

THIS REACTION FORCES ABSORBS ENERGY AT THE RATE

$$\bar{F}_R \cdot \bar{v} = -\frac{dU_{RADIATION}}{dt} = -\frac{2}{3}\frac{e^2 \dot{\bar{v}}^2}{c^3} \qquad (v \ll c)$$

ACCORDING TO LARMOR.

LORENTZ NOTED THAT IF THE ACCELERATION LASTS FROM t_1 TO t_2, WE CAN WRITE

$$\int_{t_1}^{t_2} \bar{F}_R \cdot \bar{v}\, dt = -\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2} \dot{\bar{v}} \cdot \dot{\bar{v}}\, dt$$

$$= -\frac{2}{3}\frac{e^2}{c^3}\left[\bar{v}\cdot\dot{\bar{v}}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \bar{v}\cdot\ddot{\bar{v}}\, dt \right]$$

IF $t_2 - t_1$ IS FINITE, OR IF THE MOTION IS OSCILLATORY, WE MAY SAFELY NEGLECT THE FIRST TERM.

$$\int_{t_1}^{t_2} \left(\bar{F}_R - \frac{2}{3}\frac{e^2 \ddot{\bar{v}}}{c^3} \right) \cdot \bar{v}\, dt = 0$$

HENCE WE IDENTIFY $\bar{F}_{REACTION} = \frac{2}{3}\frac{e^2 \ddot{\bar{v}}}{c^3} = \frac{2}{3}\frac{e^2 \dddot{\bar{x}}}{c^3}$ 3RD DERIVATIVE

THE EQUATION OF MOTION IS THUS

$$\bar{F}_{EXT} + \bar{F}_R = (M + M_{EL})\ddot{x}$$

OR $M_{TOT}\ddot{\bar{x}} = \bar{F}_{EXT} + \frac{2}{3}\frac{e^2 \dddot{\bar{x}}}{c^3}$

THE RADIATION REACTION TERM DOES NOT INVOLVE THE RADIUS OF THE POINT CHARGE, AND SEEMS MUCH MORE SOUNDLY MOTIVATED THAN THE ELECTROMAGNETIC MASS TERM.

NONETHELESS IT TOO HAS ITS PECULIARITIES!

SUPPOSE $\bar{F}_{EXT} = 0$. THEN $\dddot{\bar{X}} = \dfrac{3mc^3}{2e^2}\ddot{\bar{X}} = \dfrac{d}{dt}\left(\ddot{\bar{X}}\right)$

THIS HAS THE <u>RUNAWAY</u> SOLUTION $\ddot{\bar{X}} = \ddot{\bar{X}}_0 \, e^{\frac{3}{2}\frac{c}{r_0}t}$

WHERE $r_0 = e^2/mc^2$ = CLASSICAL ELECTRON RADIUS

THE CHARACTERISTIC BLOW-UP TIME IS $\tau_0 = \dfrac{2r_0}{3c} \sim 10^{-23}$ SECONDS!

TO GET RID OF THE RUNAWAY SOLUTIONS PEOPLE HAVE MADE CONTORTIONS OF A SPECTACULAR NATURE, WHICH WE WILL NOT DESCRIBE HERE. BUT WE WILL SUGGEST THAT THIS PATHOLOGY INDICATES THAT THERE WILL BE TROUBLE WITH OUR CLASSICAL MODEL OF A POINT CHARGE IN SITUATIONS WHERE

$|F_{EXT}| \lesssim |\bar{F}_R|$

TO GET AN IDEA OF THIS LIMITATION WE SUPPOSE THE EXTERNAL FORCE IS DUE TO EXTERNAL ELECTROMAGNETIC FIELDS.

THEN $m\ddot{\bar{x}} = e\bar{E} + e\dfrac{\bar{v}}{c}\times\bar{B} \underbrace{+\dfrac{2e^2}{3}\dfrac{\dddot{\bar{X}}}{c^3}}_{\text{HOPEFULLY SMALL}}$

SO $\dddot{\bar{X}} \sim \dfrac{e\dot{\bar{E}}}{m} + \dfrac{e}{m}\dfrac{\dot{\bar{v}}}{c}\times\bar{B} + \dfrac{e}{m}\dfrac{\bar{v}}{c}\times\dot{\bar{B}}$

FURTHER SUPPOSING THAT $\bar{v}/c$ IS SMALL

$\ddot{\bar{x}} = \dot{\bar{v}} \sim \dfrac{e\bar{E}}{m}$ SO $\dddot{\bar{X}} \sim \dfrac{e\dot{\bar{E}}}{m} + \dfrac{e^2}{m^2c}\bar{E}\times\bar{B}$

$\bar{F}_{REACTION} \sim \dfrac{2}{3}\dfrac{e^2}{c^3}\left(\dfrac{e}{m}\dot{\bar{E}} + \dfrac{e^2}{m^2c}\bar{E}\times\bar{B}\right)$

FOR OSCILLATORY EXTERNAL FIELDS, $\bar{E} = \bar{E}_0 e^{-i\omega t}$, $\dot{\bar{E}} \sim \omega\bar{E}$

SO $\bar{F}_{REACTION} \sim \dfrac{2}{3}\dfrac{e^2}{mc^2}\dfrac{\omega}{c}eE + \dfrac{2}{3}\left(\dfrac{e^2}{mc^2}\right)^2\dfrac{eEB}{e}$

$\sim F_{EXT}\left(\dfrac{r_0}{\lambda} + \dfrac{r_0^2 B}{e}\right)$

HENCE WE EXPECT BREAKDOWNS IN CLASSICAL E & M

IF $x \lesssim r_0 \sim 10^{-13}\,\text{cm} = \frac{e^2}{mc^2} \cdot \frac{e^2}{\hbar c}\frac{\hbar}{mc} = \alpha\,\lambda_{\text{COMPTON}} \ll \lambda_{\text{COMPTON}}$

OR $E, B \gtrsim \frac{e}{r_0^2} = \frac{m^2c^4}{e^3} = \frac{m^2c^3}{e\hbar}\frac{\hbar c}{e^2} = \frac{E_{\text{CRIT}}}{\alpha} \gg E_{\text{CRIT}}.$

[NOTE THAT THESE REMARKS DON'T DEPEND ON THE SIZE OF $\hbar$, ONLY THAT IT EXISTS!]

IN EITHER CASE ONE FINDS THE NEED FOR QUANTUM ELECTRODYNAMICS; I.E. WE ARE OUTSIDE THE VALIDITY OF CLASSICAL PHYSICS.

UNIFORMLY ACCELERATED MOTION

[F. ROHRLICH 'CLASSICAL CHARGED PARTICLES'
J.D. HAMILTON, AM. J. PHYS 46, 83 (1978)
J. COHN AM. J. PHYS 46, 225 (1978)
D.G. BOULWARE ANN. PHYS 124, 169 (1980)]

RETURNING TO THE EXPRESSION

$$\bar{F}_{\text{REACTION}} = \frac{2}{3}\frac{e^2\dddot{\bar{x}}}{c^3}$$

ONE OF THE SIMPLEST APPLICATIONS MIGHT BE TO THE CASE OF UNIFORMLY ACCELERATED MOTION : $\ddot{\bar{x}}$ = CONSTANT

IF SO, $\bar{F}_R = 0$!

IN THE EARLY DAYS THIS LED MANY FAMOUS PEOPLE TO CONCLUDE THAT A UNIFORMLY ACCELERATED CHARGE DOES NOT RADIATE! THIS CERTAINLY WOULD BE A BLOW FOR ALL THE EFFORT WE SPENT IN TRYING TO UNDERSTAND RADIATION.

TO OUR RELIEF THE MODERN CONSENSUS IS THAT INDEED A UNIFORMLY ACCELERATED CHARGE RADIATES, AND THAT THE RADIATION REACTION IS ALSO ZERO. THIS APPARENT CONTRADICTION IS RESOLVED BY A DETAILED EXAMINATION OF THE FIELD ENERGY. TYPICALLY WE WOULD SAY THAT

$$U_{\text{FIELD}} = U_{\text{RADIATION}} + U_{\text{NEAR}}$$

WHERE U_{NEAR} IS THAT ENERGY WHICH IS TIED TO THE MOVING CHARGE.

WHAT APPEARS TO HAPPEN FOR UNIFORMLY ACCELERATED MOTION IS THAT U_{FIELD} = CONSTANT , BUT $U_{\text{RADIATION}}$ GROWS WITH TIME WHILE U_{NEAR} SHRINKS. THEN NO MECHANICAL ENERGY GOES INTO CHANGING U_{FIELD}, AS WOULD NORMALLY BE EXPECTED.

SOME PLAUSIBILITY MAY BE LENT TO THESE STATEMENTS IF WE EXAMINE THE CHARACTER OF UNIFORMLY ACCELERATED MOTION.

FIRST WE MUST DEFINE WHAT IS MEANT BY 'UNIFORM ACCELERATION' IN SPECIAL RELATIVITY. WE REQUIRE THAT THE ACCELERATION AS

VIEWED IN THE INSTANTANEOUS REST FRAME OF THE PARTICLE BE CONSTANT.

FOR MOTION ALONG A STRAIGHT LINE THIS LEADS TO FAIRLY SIMPLE RELATIONS. THEN $\bar{a}$ IS $\parallel$ TO $\bar{v}$ AND SO

$$a^{*2} = -c^4 a_\mu a^\mu = \gamma^6 a^2 \qquad \text{(p221 LECTURE 18)}$$

SO $\gamma^3 a = a^* =$ CONSTANT

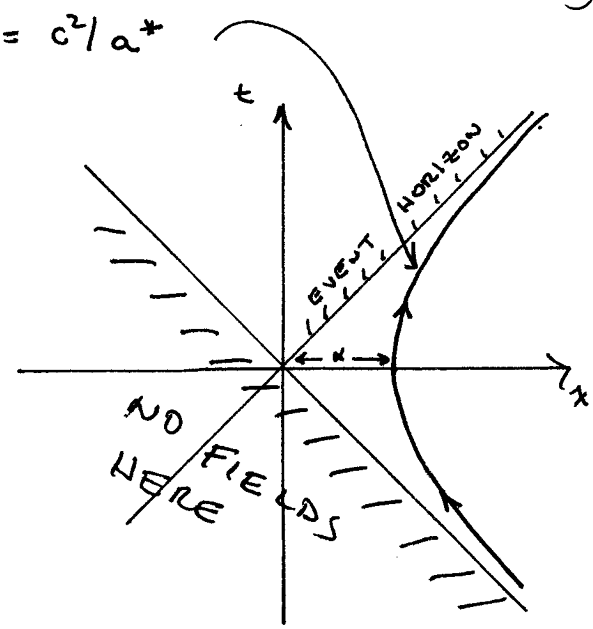
NOW $\gamma^3 a = \dfrac{d\,\gamma v}{dt} \Rightarrow \gamma v = a^* t$

$$\Rightarrow v = \frac{dx}{dt} = \frac{c a^* t}{\sqrt{c^2 + (a^* t)^2}} \qquad \left[\text{USING } \gamma = \sqrt{1+\gamma^2\beta^2}\right]$$

$$\Rightarrow x = \frac{c}{a^*}\sqrt{c^2 + (a^* t)^2} \qquad \left(+\text{ CONST.} = -\alpha \text{ IF WANT } x(0)=0\right)$$

OR $\underline{x = \sqrt{\alpha^2 + c^2 t^2}}$ WHERE $\alpha = c^2/a^*$

THIS IS A HYPERBOLA, AND SO UNIFORMLY ACCELERATED MOTION IS SOMETIMES CALLED "HYPERBOLIC MOTION."



AS $t \to \pm\infty$ $v \to \pm c$ WHICH LENDS A SPECIAL CHARACTER TO THIS MOTION.

FOR EXAMPLE, THE RADIATION FIELDS MOVE WITH VELOCITY c, SO AS $t \to \pm\infty$ THE RADIATION CAN HARDLY BE DISTINGUISHED FROM THE NEAR FIELD!

ALSO NOTE THAT THERE CAN BE NO FIELDS AT ALL FROM THE CHARGE IN THE LOWER LEFT HALF SPACE AS SHOWN. ON THE OTHER HAND, ALONG THE BOUNDARY OF THIS HALF SPACE THERE IS A 'SHOCK WAVE' DISCONTINUITY DUE TO THE TREMENDOUS AMOUNT OF RADIATION FROM $t \sim -\infty$.

SIMILARLY, NO OBSERVER IN THE UPPER LEFT HALF SPACE CAN COMMUNICATE WITH THE MOVING CHARGE! THE LINE $x = t$ IS SAID TO BE AN 'EVENT HORIZON' FOR THE CHARGE. THE CONCEPT OF 'EVENT HORIZON' IS MOST OFTEN HEARD OF IN CONJUNCTION WITH 'BLACK HOLES'....

SOME FINAL TITILLATIONS INVOLVE THE PRINCIPLE OF EQUIVALENCE.

1. WE CERTAINLY AGREE THAT AN OBSERVER AT REST WATCHING ANOTHER CHARGE AT REST WILL SEE NO RADIATION.

2. IF THE CHARGE IS ACCELERATED AND THE OBSERVER IS AT REST THEN RADIATION IS OBSERVED.

3. SUPPOSE THE CHARGE IS AT REST, BUT THE OBSERVER IS ACCELERATED. WHAT DOES (S)HE SEE? RADIATION!

4. SUPPOSE BOTH THE CHARGE AND OBSERVER ARE ACCELERATED, SO THAT THERE VELOCITIES ARE ALWAYS EQUAL (IN SOME LAB FRAME) NOW THE OBSERVER SEES NO RADIATION!

EINSTEIN HAS REVEALED HOW AN OBSERVER AT REST IN A UNIFORM GRAVITATIONAL FIELD SHOULD FIND THE SAME PHYSICAL PHENOMENA AS A UNIFORMLY ACCELERATED OBSERVER.

MORE BASICALLY, HE NOTED THAT AN OBSERVER FALLING FREELY IN A GRAVITATIONAL FIELD WILL THINK HE IS IN AN INERTIAL FRAME!

CAN WE RELATE RADIATION FROM ACCELERATED CHARGES TO RADIATION FROM CHARGES IN A GRAVITATIONAL FIELD? CERTAINLY!

(SOME PEOPLE HAVE SAID NO, BECAUSE THE PRINCIPLE OF EQUIVALENCE IS A <u>LOCAL</u> PRINCIPLE, WHILE RADIATION IMPLIES ACTIVITY THAT IS REMOVED FROM THE LOCALE OF THE OBSERVER. THIS OBJECTION IS SPURIOUS.)

BUT WE MUST BE CAREFUL! THE TRICK QUESTION IS WHETHER WE SHOULD EXPECT A CHARGE SITTING IN FRONT OF US TO RADIATE JUST BECAUSE IT IS IN A UNIFORM GRAVITATIONAL FIELD? (IF SO WE HAVE A WAY OF DETECTING GRAVITY...)

THERE IS NO DIFFICULTY IF WE SPECIFY THE RELATIVE ROLE OF CHARGE AND OBSERVER.

5. IF BOTH CHARGE AND OBSERVER ARE AT REST IN A GRAVITATIONAL FIELD, THEY ARE EQUIVALENT TO BOTH BEING ACCELERATED. THIS IS CASE 4 $\Longrightarrow$ NO RADIATION.

6. LIKEWISE IF BOTH CHARGE AND OBSERVER ARE FALLING FREELY, WE EXPECT NO RADIATION.

7.8. BUT IF ONE IS AT REST AND THE OTHER FALLING FREELY, RADIATION IS OBSERVED!

A FOOTNOTE ON ANGULAR MOMENTUM

WE HAVE PREVIOUSLY REMARKED THAT THE RADIATION FIELDS CAN CARRY AWAY ANGULAR MOMENTUM. WE CAN ACCOUNT FOR THE CONSEQUENT LOSS OF ANGULAR MOMENTUM AT THE SOURCE BY MEANS OF THE TORQUE DUE TO THE RADIATION REACTION FORCE.

NAMELY $$\frac{d\bar{L}}{dt} = \bar{r} \times \bar{F}_R = \frac{2}{3}\frac{e^2}{c^3}\,\bar{r} \times \dddot{\bar{r}} = \frac{2}{3c^3}\left[e\bar{r} \times e\dddot{\bar{r}}\right]$$

BUT WE NOTE THAT $e\bar{r} = \bar{p}$ = DIPOLE MOMENT

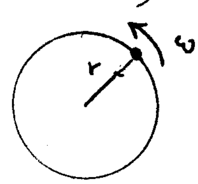
[IF SEVERAL CHARGES ARE PRESENT WE HAVE A SUM OF TERMS AS ABOVE. WE CAN STILL INTRODUCE $\bar{p}$ = TOTAL DIPOLE MOMENT.]

THUS $$\frac{d\bar{L}}{dt} = \frac{2}{3c^3}\,\bar{p} \times \dddot{\bar{p}} = \frac{2}{3c^3}\left[\frac{d}{dt}\left(\bar{p} \times \ddot{\bar{p}}\right) - \dot{\bar{p}} \times \ddot{\bar{p}}\right]$$

AS FOR THE ENERGY LOSS RELATION, WE OBTAIN A SIMPLER FORM IF WE AVERAGE OVER TIME. IN PARTICULAR, FOR <u>PERIODIC</u> <u>MOTION</u>, THE AVERAGE OF THE FIRST TERM VANISHES. (IF WE IGNORE THE SLOW DECREASE IN $\ddot{\bar{p}}$ DUE TO THE ENERGY LOSS!)

SO $$\left\langle \frac{d\bar{L}}{dt} \right\rangle = -\frac{2}{3c^3}\left\langle \dot{\bar{p}} \times \ddot{\bar{p}} \right\rangle$$

<u>EXAMPLE</u> POINT CHARGE MOVING IN A CIRCLE (PROB 3, SET 8)



$$\bar{p} = er\,(\hat{x} + i\hat{y})\,e^{-i\omega t}$$

$$\dot{\bar{p}} = -i\omega\,\bar{p} \qquad \ddot{\bar{p}} = -\omega^2\,\bar{p}$$

$$\left\langle \frac{d\bar{L}}{dt} \right\rangle = -\frac{2\omega^3}{3c^3}\,\frac{\mathrm{Re}\left(i\,\bar{p} \times \bar{p}^*\right)}{2}$$

BY AN EXTENSION OF OUR TRICKS FOR THE TIME AVERAGE OF COMPLEX QUANTITIES

$$= -\frac{\cancel{2}\,\omega^3}{3\,c^3}\,p_0^2\,\hat{z}$$

$[p_0 \equiv er]$

$$= -\frac{1}{\omega}\left(\frac{2}{3}\frac{\omega^4 p_0^2}{c^3}\right)\hat{z}$$

$$= \frac{\langle dU_{LOST}/dt\rangle}{\omega}\,\hat{z}$$

AS FOUND ON THE PROBLEM SET!