STEADY CURRENTS

WE NOW REMOVE THE RESTRICTION THAT THE ELECTRICAL CHARGES BE AT REST, AND CONSIDER THE NEW PHENOMENA WHICH ARISE WHEN ELECTRICAL CURRENTS ARE PRESENT.

INITIALLY WE SUPPOSE THAT THE CURRENTS ARE STEADY, AS DEFINED BELOW. EVEN IN THIS CASE WE HAVE 2 CLASSES OF PHENOMENA TO CONSIDER:

— THE ELECTRICAL PROPERTIES OF THE STEADY CURRENTS
— MAGNETISM CAUSED BY THE CURRENTS.

CURRENT DENSITY (BECKER SEC. 38)

IN ADDITION TO THE TOTAL CURRENT $I = dQ/dt$ FLOWING THRU SOME CONDUCTOR, WE INTRODUCE THE CURRENT DENSITY $\bar{J}$

BY THE RELATION $I = \int_{\text{SURFACE}} \bar{J}\cdot d\bar{S}$ = CURRENT THRU THE SURFACE.

DIMENSIONALLY: $[\bar{J}] = \dfrac{[\text{CHARGE}]}{[\text{AREA}][\text{TIME}]} = \dfrac{[\text{CHARGE}]}{[\text{VOLUME}]}\cdot[\text{VELOCITY}]$

$\bar{J}$ IS A CONTINUOUS FUNCTION DESCRIBING THE MOTION OF THE CHARGE DENSITY ρ. IF ALL THE CHARGE HAD VELOCITY $\bar{v}$, THEN $\bar{J} = \rho\bar{v}$ ETC.

EMPIRICALLY IT IS OBSERVED THAT CHARGE IS CONSERVED. IF WE WATCH THE CHARGE INSIDE A VOLUME, IT CAN ONLY CHANGE IF SOME CHARGE FLOWS ACROSS THE SURFACE.

I.e. $$\frac{d}{dt}\int_{\text{VOLUME}} \rho \, d\text{vol} = -\int_{\text{SURFACE}} \bar{J}\cdot d\bar{S}$$ (− IF $d\bar{S}$ POINTS ALONG THE OUTWARD NORMAL)

BY GAUSS' LAW THIS BECOMES $$\int \left(\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot\bar{J}\right) d\text{vol} = 0$$

SO $$\underline{\bar{\nabla}\cdot\bar{J} + \frac{\partial\rho}{\partial t} = 0}$$ EXPRESSES THE FACT OF CHARGE CONSERVATION

IN STEADY CURRENT SITUATIONS WE SUPPOSE THAT

$\dfrac{\partial\rho}{\partial t} = 0$ AS WELL AS $\dfrac{\partial\bar{J}}{\partial t} = 0$. THEN WE OBTAIN

THE ADDITIONAL RESTRICTION THAT $\underline{\bar{\nabla}\cdot\bar{J} = 0}$, $\Rightarrow$ STEADY CURRENTS FLOW IN CLOSED LOOPS.

PH 206 LECTURE 7

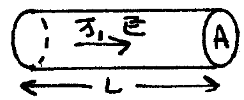
OHM'S LAW (BECKER SEC 39)

IN MANY PRACTICAL CASES THERE IS A SIMPLE RELATION BETWEEN THE CURRENT DENSITY AND THE ELECTRIC FIELD:

$$\bar{J} = \sigma \bar{E} \qquad \text{WHERE } \sigma = \underline{\text{CONDUCTIVITY}}$$

σ IS A CONSTANT OF THE MATERIAL UNDER CONSIDERATION.

WE RECOVER THE USUAL FORM OF OHM'S LAW BY CONSIDERING A VOLUME OF MATERIAL OF LENGTH L ALONG THE DIRECTION OF $\bar{E}$ AND AREA A $\perp$ TO $\bar{E}$.



THEN THE TOTAL CURRENT IS $I = JA$, WHILE THE POTENTIAL DROP ACROSS THE VOLUME IS $V = \int \bar{E}\cdot d\bar{l} = EL$

$$\text{SO} \quad V = \frac{JL}{\sigma} = I\frac{L}{\sigma A}$$

HENCE IF WE DEFINE $R = \frac{L}{\sigma A}$ = <u>RESISTANCE</u> OF THE VOLUME

THEN $V = IR$ AS USUAL. NOTE THAT THE DEFINITION OF RESISTANCE IS VERY DEPENDENT ON GEOMETRY, WHILE σ IS NOT.

WE RECALL THE BASIC FEATURES OF THE <u>DRUDE MODEL OF CONDUCTIVITY</u>. (1900)

INSIDE THE MATERIAL THERE ARE n ELECTRONS PER UNIT VOLUME WHICH ARE FREE TO MOVE UNDER THE INFLUENCE OF AN APPLIED ELECTRIC FIELD. THESE ELECTRONS ARE IN THERMAL MOTION LIKE THAT OF MOLECULES IN A GAS AT TEMPERATURE T. THE ELECTRONS COLLIDE WITH THE FIXED MATERIAL QUITE OFTEN, AND WE LABEL τ = MEAN FREE TIME BETWEEN COLLISIONS. DURING TIME τ, AN ELECTRON AQUIRES VELOCITY INCREMENT

$$\Delta v = \frac{F}{m}\tau = \frac{eE}{m}\tau$$

SUPPOSING THE COLLISIONS HAVE THE EFFECT OF RESETTING THE AVERAGE VELOCITY TO ZERO, WE HAVE AN AVERAGE 'DRIFT VELOCITY'

$$v_{\text{DRIFT}} = \frac{1}{2}\frac{eE}{m}\tau$$

THE CURRENT DENSITY IS THEN $J = \underbrace{ne}_{\rho} v_{\text{DRIFT}} = \frac{ne^2\tau}{2m}E$

$$\text{SO} \quad \sigma = \frac{ne^2\tau}{2m}$$

JOULE HEATING (BECKER SEC. 40)

CURRENTS IN A REGION OF NON-ZERO CONDUCTIVITY ARE NOT SELF-SUSTAINING. ENERGY MUST BE PROVIDED TO MAINTAIN THEM.

SUPPOSE CHARGE dQ IS TRANSPORTED THRU POTENTIAL DIFFERENCE $d\phi$ IN TIME dt DUE TO THE CURRENT. THEN THE POWER REQUIRED IS

$$\frac{dU}{dt} = \frac{dQ\, d\phi}{dt} = \frac{dQ}{dA\,dt}\,\frac{d\phi}{d\ell}\,dA\,d\ell = \bar{J}\cdot\bar{E}\,d\mathrm{vol}.$$

THE ENERGY LOST IS DISSAPATED IN THERMAL MOTION DUE TO THE COLLISIONS OF THE CONDUCTION ELECTRONS WITH THE FIXED MATTER IN THE CONDUCTOR. AS SUCH, WE MAY DEFINE THE POWER DENSITY

$$\frac{dU}{d\mathrm{vol}\,dt} = \bar{J}\cdot\bar{E} = \sigma E^2 = \frac{J^2}{\sigma}, \quad \text{IF } \bar{J} = \sigma\bar{E}.$$

THE INCREASED THERMAL MOTION OF THE CONDUCTOR IS DESCRIBED AS HEATING BY THE MACROSCOPIC OBSERVER. THIS IS THE JOULE HEATING PHENOMENON, DESCRIBED BY I^2R UPON INTEGRATION OVER THE ENTIRE VOLUME.

IF THE ONLY SOURCE OF ENERGY IS THAT STORED IN THE ELECTRIC FIELD, $U_{EL} = \frac{1}{8\pi}\int \bar{E}\cdot\bar{D}\,d\mathrm{vol}$, THEN THE JOULE HEATING WOULD SOON CAUSE THE ELECTRIC FIELD ENERGY TO VANISH $\Rightarrow$ NO MORE CURRENT, SINCE $\bar{J} = \sigma\bar{E}$. HENCE STEADY CURRENTS CANNOT BE MAINTED BY AN ELECTROSTATIC CHARGE DISTRIBUTION ALONE!

EXAMPLE RELAXATION OF THE FIELD IN A CONDUCTOR.

SUPPOSE AT SOME MOMENT WE HAVE $\bar{E} \neq 0$ AND $\rho_{FREE} \neq 0$ INSIDE A CONDUCTOR, AND THERE IS NO OUTSIDE SOURCE OF ENERGY TO MAINTAIN THIS SITUATION.

THEN $\bar{J} = \sigma\bar{E} = \frac{\sigma}{\epsilon}\bar{D}$ WHERE ϵ = DIELECTRIC CONSTANT

CHARGE CONSERVATION $\Rightarrow$ $\frac{\partial \rho_{FREE}}{\partial t} = -\bar{\nabla}\cdot\bar{J} = -\frac{\sigma}{\epsilon}\bar{\nabla}\cdot\bar{D} = -\frac{4\pi\sigma}{\epsilon}\rho_{FREE}$

HENCE, $\rho_{FREE} = \rho_0\, e^{-\frac{4\pi\sigma}{\epsilon}t}$.

THUS ρ_{FREE}, AND HENCE $\bar{E}$, VANISH RAPIDLY INSIDE A CONDUCTOR WHEN THE EXTERNAL SITUATION IS 'ELECTROSTATIC'!

IN CGS UNITS, TYPICAL METALS HAVE CONDUCTIVITIES $\sim 10^{17}$ $\Rightarrow$ RELAXATION TIME $\sim 10^{-18}$ SEC! (THIS TIME IS PERHAPS TOO SHORT. SEE H.C. OHANIAN, AM.J.PHYS 51, 1020 (1983)

Ph 206 LECTURE 7

ELECTROMOTIVE FORCES (EMF) (BECKER SEC 41)

AN ENERGY SOURCE OUTSIDE THE SCOPE OF ELECTROSTATICS IS REQUIRED TO MAINTAIN STEADY CURRENTS. EXAMPLES ARE BATTERIES AND GENERATORS. SUCH ENTITIES HAVE BEEN SAID TO POSSESS THE ELECTROMOTIVE FORCE WHICH DRIVES THE CURRENT.

WE SKETCH HOW WE MIGHT SLIGHTLY EXTEND OUR DISCUSSION OF ELECTROSTATICS TO INCLUDE THE EMF.

WE SUPPOSE THAT OUTSIDE THE SOURCE OF THE EMF, THE ELECTRIC FIELD IS ELECTROSTATIC, AND OBEYS

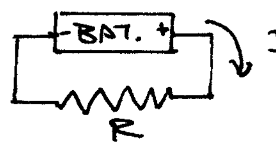
$$\bar{\nabla} \times \bar{E} = 0 \qquad \bar{\nabla} \cdot \bar{E} = 4\pi \rho_{TOTAL}$$

(WE MAY CONTINUE OUR CONCEPTION OF THE ELECTROSTATIC FIELD INSIDE THE SOURCE OF THE EMF BY OUR USUAL METHOD OF SOLVING BOUNDARY VALUE PROBLEM. THEN, $\oint_{LOOP} \bar{E} \cdot d\bar{l} = 0$.)

INSIDE THE SOURCE WE SUPPOSE THERE IS AN ADDITIONAL, NON-ELECTROSTATIC FIELD $\bar{E}'$ SUCH THAT

$$\bar{J} = \sigma(\bar{E} + \bar{E}')$$

CONSIDER A SIMPLE CIRCUIT:



WE SAY THAT $EMF = \mathcal{E} = \oint_{LOOP} \bar{E}' \cdot d\bar{l} = \oint_{LOOP} (\bar{E} + \bar{E}') \cdot d\bar{l} = \oint \frac{\bar{J} \cdot d\bar{l}}{\sigma} = \frac{IL}{\sigma A} = IR$

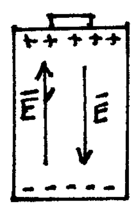
THUS THE IDEA OF A NON-ELECTROSTATIC FIELD INSIDE THE SOURCE ALLOWS US TO FORMULATE THE USUAL KIRCHOFF CIRCUIT LAW RELATING EMF TO CURRENT AND RESISTANCE.

NOTE THAT $\oint_{CLOSED\ LOOP} \bar{E}' \cdot d\bar{l} = \mathcal{E} \neq 0$ SO $\bar{\nabla} \times \bar{E}' \neq 0$

$\Rightarrow$ $\bar{E}'$ CANNOT BE DERIVED FROM A(N ELECTROSTATIC) POTENTIAL.

TO FURTHER TEST OUR CONCEPTION, CONSIDER A CIRCUIT CONTAINING AN EMF WHICH IS NOT COMPLETE — i.e., A BATTERY BY ITSELF. THEN $\bar{J} = 0 \Rightarrow \bar{E} = -\bar{E}'$

THE NEW FIELD $\bar{E}'$ PUSHES CHARGES TO THE + TERMINAL OF THE BATTERY. THEN AN ELECTROSTATIC FIELD $\bar{E}$ IS SET UP WHICH IS EXACTLY OPPOSITE TO $\bar{E}'$.



THIS IS CLEARLY A NON-ELECTROSTATIC SITUATION. FOR A 'DRY CELL' - A CHEMICAL PROCESS PROVIDES $\bar{E}'$ ....

Ph 206 Lecture 7

Remarks on Steady Current Problems

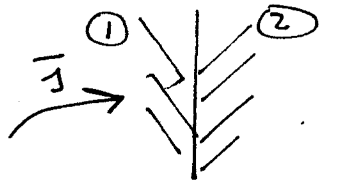
Generally the source of the EMF is confined to some finite physical region. Outside that region we have

$$\bar{J} = \sigma \bar{E} \qquad \text{where} \qquad \bar{\nabla} \times \bar{E} = 0$$

That is we still can solve for $\bar{E}$ outside the source by our techniques of electrostatics. [$\rho_{free} = 0$ inside a static conductor]

i.e. $\bar{E} = -\bar{\nabla}\phi$ and $\bar{\nabla} \cdot \bar{J} = 0 \Rightarrow \nabla^2 \phi = 0$ (Laplace's eq again)

To complete the solution we must be aware of the proper boundary conditions. A new condition at the boundary between two conductors is obtained from charge conservation: $\bar{\nabla} \cdot \bar{J} = 0 \Rightarrow$



$$\int \bar{J}_1 \cdot d\bar{s} = \int \bar{J}_2 \cdot d\bar{s} \text{ at the boundary}$$

$$\Rightarrow \sigma_1 E_{1\perp} = \sigma_2 E_{2\perp}$$

$$\bar{\nabla} \times \bar{E} = 0 \Rightarrow E_{1\parallel} = E_{2\parallel}$$

Note that this is similar to dielectric boundaries if we identify $\bar{J} \Leftrightarrow \bar{D}$ $\quad \sigma \Leftrightarrow \epsilon$.

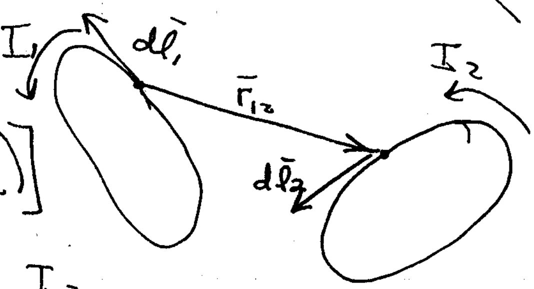
Magnetostatics (Becker sec 46)

As discussed in lecture 1, the early investigations into magnetism by Coulomb and others utilized those substances known as permanent magnets. All such magnets contained two opposite 'poles', which appeared to obey laws very similar to those for electricity:

$$\bar{F} \sim \frac{p_1 p_2}{r^2} \hat{r} + \ldots$$

A tremendous advance was made in 1820 by Oersted who discovered that current carrying wires displayed many properties similar to those of permanent magnets. In the next decade Ampere provided a thorough analysis of the magnetic effects of current loops. A prominent feature of this analysis is the hypothesis that all magnetic effects are due to currents. Permanent magnets are then to have some kind of permanent internal currents. In any case, Ampere unified the study of 'magnetostatics' with that of steady currents.

AMPÈRE (1820-23) WROTE



$$\bar{F}_{ON\,2} = \frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{\hat{r}_{12}}{r_{12}^2} \left[3(d\bar{l}_1\cdot\hat{r})(d\bar{l}_2\cdot\hat{r}) - 2(d\bar{l}_1\cdot d\bar{l}_2)\right]$$

FOR THE FORCE ON LOOP 2 CARRYING CURRENT I_2,

DUE TO CURRENT I_1 IN LOOP 1. C IS A CONSTANT, 3×10^{10} cm/s (A VELOCITY)

THIS IS A $1/r^2$ FORCE LAW, BUT WITH A COMPLICATED ANGULAR DEPENDENCE.

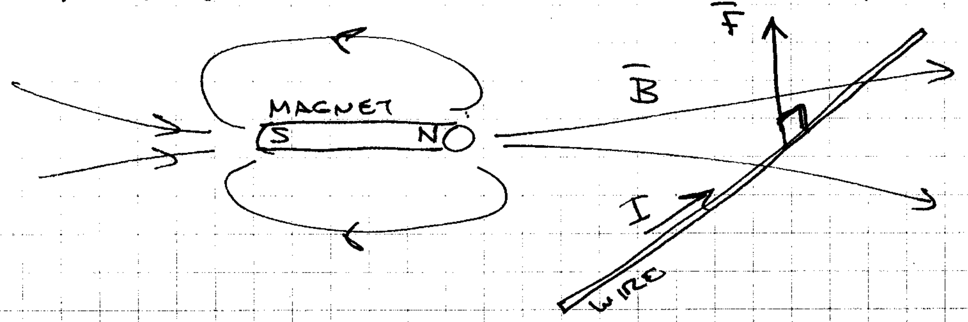
NOTE THAT $d\bar{F}_{ON\,2} = -d\bar{F}_{ON\,1}$, SINCE ONLY $\hat{r}_{12}$ CHANGES SIGN IF SWAP 1 & 2

$\Rightarrow \bar{F}_{ON\,2} = -\bar{F}_{ON\,1} \Leftrightarrow$ OBEYS NEWTON'S 3RD LAW; $d\bar{F}$ IS ALONG LINE OF CENTERS.

AMPÈRE'S FORCE LAW DOES NOT FACTORIZE: $F_{ON\,2} \neq \underbrace{\oint_1 \cdots}_{\text{PROPERTY OF CIRCUIT 1}} \underbrace{\oint_2 \cdots}_{\text{EFFECT OF CIRCUIT 2 AT A POINT IN CIRCUIT 1 (i.e. A FIELD)}}$

AMPÈRE CLAIMED THAT MAGNETISM IS ACTUALLY DUE TO CURRENTS.

BUT EXPERIMENTS WITH MAGNETS + CIRCUITS SHOWED THAT $\bar{F}$ IS NOT ALONG THE LINE OF CENTERS BETWEEN MAGNET & CIRCUIT.



BIOT-SAVART (1820) STUDIED MAGNETS + CIRCUITS AND PROPOSED THAT

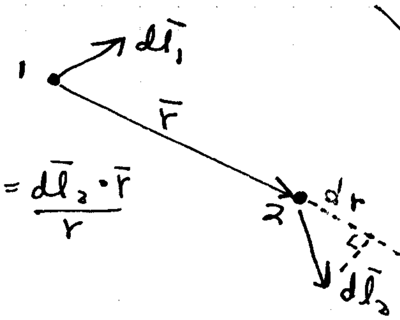
$$\bar{F}_{ON\,CIRCUIT} = \frac{I}{c}\oint d\bar{l} \times \bar{B}_{MAGNET}$$

ARE THE CLAIMS OF AMPÈRE, BIOT & SAVART CONSISTENT? YES! AMPERE (1825) GRASSMANN (1844)

IDEA: $\oint_2$ NOT CHANGED IF ADD ANY PERFECT DIFFERENTIAL (FOR FIXED POINT IN LOOP 1)

i.e., $\oint_2 d\left[f(1,2)\right]_{1\,FIXED} = 0$

Ph 206 Lecture 7



IN PARTICULAR, CONSIDER $d\left[\bar{r}\,(d\bar{l}_1\cdot\bar{r})\,f(r)\right)$

NOTE THAT FOR FIXED POINT 1, $d\bar{l}_2 = d\bar{r}$ & $dr = d\bar{r}\cdot\hat{r} = \dfrac{d\bar{l}_2\cdot\bar{r}}{r}$

SO $df = \dfrac{df}{dr}\,dr = f'\,\dfrac{(d\bar{l}_2\cdot\bar{r})}{r}$

THEN, $d\left[\bar{r}\,(d\bar{l}_1\cdot\bar{r})\,f\right] = d\bar{l}_2\,(d\bar{l}_1\cdot\bar{r})\,f + \bar{r}\,(d\bar{l}_1\cdot d\bar{l}_2)\,f + \bar{r}\,(d\bar{l}_1\cdot\bar{r})(d\bar{l}_2\cdot\bar{r})\,\dfrac{f'(r)}{r}$

LET $f(r) = \dfrac{I_1 I_2}{c^2}\cdot\dfrac{1}{r^3}$ SO $f' = -\dfrac{I_1 I_2}{c^2}\,\dfrac{3}{r^4}$

WE CAN NOW MODIFY AMPÈRES FORCE LAW TO BECOME

$$\bar{F}_{\text{ON }2} = \frac{I_1 I_2}{c^2}\oint_1\oint_2\left\{\frac{\hat{r}}{r^2}\left[3(d\bar{l}_1\cdot\hat{r})(d\bar{l}_2\cdot\hat{r}) - 2(d\bar{l}_1\cdot d\bar{l}_2)\right]\right.$$

$$\left. + \frac{d\bar{l}_2\,(d\bar{l}_1\cdot\hat{r})}{r^2} + \frac{\hat{r}\,(d\bar{l}_1\cdot d\bar{l}_2)}{r^2} - \frac{3\,\hat{r}\,(d\bar{l}_1\cdot\hat{r})(d\bar{l}_2\cdot\hat{r})}{r^2}\right\}$$

$$= \frac{I_1 I_2}{c^2}\oint_1\oint_2 \frac{\left[d\bar{l}_2\,(d\bar{l}_1\cdot\hat{r}) - \hat{r}\,(d\bar{l}_1\cdot d\bar{l}_2)\right]}{r^2}$$

$$\bar{F}_{\text{ON }2} = \frac{I_1}{c}\oint d\bar{l}_1 \times \frac{I_2}{c}\oint\frac{d\bar{l}_2\times\hat{r}}{r^2} \equiv \frac{I_1}{c}\oint d\bar{l}_1 \times \bar{B}(\text{AT 1, DUE TO 2})$$

WHERE $\underline{\bar{B} = \dfrac{I}{c}\oint\dfrac{dl\times\hat{r}}{r^2}}$ $\equiv$ MAGNETIC INDUCTION (BIOT-SAVART FORM)

IF WE GENERAL THE IDEA OF A FILAMENTARY CURRENT I TO A CONTINUOUS CURRENT DENSITY $\bar{J}$, THEN WE NOTE THAT $I\,d\bar{l} \rightarrow \bar{J}\,d\text{Vol}$

SO $\bar{F} = \dfrac{1}{c}\displaystyle\int \bar{J}\times\bar{B}\,d\text{vol}$ FOR A CLOSED CIRCUIT

AND $\bar{B} = \dfrac{1}{c}\displaystyle\int\dfrac{\bar{J}\times\hat{r}}{r^2}\,d\text{vol}$ DUE TO CURRENTS IN CLOSED LOOPS.

? CAN WE SPEAK OF A FORCE $d\bar{F}$ BETWEEN A PAIR OF ISOLATED CURRENTS ELEMENTS, $I_1 d\bar{l}_1$ & $I_2 d\bar{l}_2$?

AMPÈRE: $d\bar{F} = \dfrac{I_1 I_2}{c^2}\,\dfrac{\hat{r}_{12}}{r_{12}^2}\left[3(d\bar{l}_1\cdot\hat{r})(d\bar{l}_2\cdot\hat{r}) - 2(d\bar{l}_1\cdot d\bar{l}_2)\right]$ ← OBEYS NEWTON'S 3RD LAW

BIOT-SAVART: $d\bar{F} = \dfrac{I_1 I_2}{c^2}\,\dfrac{d\bar{l}_1\times(d\bar{l}_2\times\hat{r})}{r^2}$ ← DOES NOT OBEY NEWTON'S 3RD LAW!

Ph 206 Lecture 7

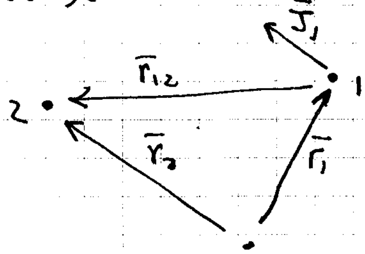
Because of it's relation to the field concept, we prefer the Biot-Savart form. To reconcile this with Newton's 3rd law, we must go beyond statics: To realize an isolated current element, we need to consider a single moving charge, $I\,d\bar{l} \to e\bar{v}$

Then $\bar{F} \to e\frac{\bar{v}}{c} \times \bar{B}$ (Lorentz force law (1890))

[or, if electric field present $F = e\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right)$.]

But need the full Maxwell equations to show that this is consistent with Newton (Poynting, Heaviside, 1884).

The Vector Potential



We can manipulate the expression

$$\bar{B}_2 = \frac{1}{c}\int_1 \frac{\bar{J}\times\hat{r}_{12}}{r_{12}^2}\,dVol_1$$

By noting that $\frac{\hat{r}_{12}}{r_{12}^2} = -\bar{\nabla}_2 \frac{1}{r_{12}}$ since $\bar{r}_{12} = \bar{r}_1 - \bar{r}_2$

So, we can write $\bar{B}_2 = \bar{\nabla}_2 \times \frac{1}{c}\int_1 \frac{\bar{J}_1\,dvol_1}{r_{12}}$

We define $\bar{A} = \frac{1}{c}\int \frac{\bar{J}\,dVol}{r} \equiv$ vector potential

Then $\underline{\bar{B} = \bar{\nabla}\times\bar{A}}$ $\Rightarrow$ $\underline{\bar{\nabla}\cdot\bar{B} = \bar{\nabla}\cdot(\bar{\nabla}\times\bar{A}) = 0}$

$\bar{\nabla}\cdot\bar{B} = 0$ is the 3rd Maxwell equation.

Gauss: $\int(\bar{\nabla}\cdot\bar{B})\,dVol = \int \bar{B}\cdot d\bar{A}_{Area} = 0$

$\Rightarrow$ No point sources of $\bar{B}$, the magnetic field. (No magnetic monopoles)

$\Rightarrow$ Lines of $\bar{B}$ always form closed loops.

This conclusion is ultimately an experimental observation. Our derivation began with the assumption of current loops, with no mention of magnetic poles. It is a step beyond logical deduction to say that all magnetic phenomena are described by Ampére's laws. However, these laws are consistent with the observation that permanent magnets always have 2 (or more) poles.

AMPERE'S LAW

WE COMPLETE THE FORMALISM OF MAGNETOSTATICS BY OBTAINING THE DIFFERENTIAL RELATION FOR $\bar{\nabla} \times \bar{B}$.

THEN THE USE OF THIS RELATION, COMBINED WITH $\bar{\nabla} \cdot \bar{B} = 0$ WILL PROVIDE AN ALTERNATIVE TO CALCULATION OF $\bar{B}$ VIA THE BIOT-SAVART INTEGRAL.

WE START WITH THE RELATION

$$\bar{B}_2 = \bar{\nabla}_2 \times \bar{A}_2 = \bar{\nabla}_2 \times \frac{1}{c} \int_1 \frac{\bar{J}_1 \, dvol_1}{r_{12}}$$

$$\bar{\nabla}_2 \times \bar{B}_2 = \bar{\nabla}_2 \times (\bar{\nabla}_2 \times \bar{A}_2) = \bar{\nabla}_2 (\bar{\nabla}_2 \cdot \bar{A}_2) - \nabla_2^2 \bar{A}_2$$

$$= \frac{1}{c} \bar{\nabla}_2 \int_1 \bar{J}_1 \cdot \bar{\nabla}_2 \left(\frac{1}{r_{12}}\right) dvol_1 - \frac{1}{c} \int_1 \bar{J}_1 \nabla_2^2 \left(\frac{1}{r_{12}}\right) dvol_1$$

$$= -\frac{1}{c} \bar{\nabla}_2 \int_1 \bar{J}_1 \cdot \bar{\nabla}_1 \left(\frac{1}{r_{12}}\right) dvol_1 + \frac{4\pi}{c} \int \bar{J}_1 \, \delta(r_{12}) \, dvol_1$$

BY PARTS

$$= +\frac{1}{c} \bar{\nabla}_2 \int_1 \frac{\bar{\nabla}_1 \cdot \bar{J}_1}{r_{12}} dvol_1 + \frac{4\pi \bar{J}_2}{c}$$

IN MAGNETOSTATICS, $\bar{\nabla} \cdot \bar{J} = 0$, SO

BUT IN GENERAL WE MIGHT EXPECT AN ADDITIONAL TERM IN THE $\bar{\nabla} \times \bar{B}$ EQUATION!

$$\underline{\bar{\nabla} \times \bar{B} = \frac{4\pi \bar{J}}{c}} \qquad \text{AMPERE'S LAW}$$

THIS IS OFTEN EXPRESSED IN INTEGRAL FORM!

$$\int_{\substack{\text{SURFACE}\\ \text{OF LOOP}}} \bar{\nabla} \times \bar{B} \cdot d\bar{S} = \frac{4\pi}{c} \int \bar{J} \cdot d\bar{S} = \frac{4\pi}{c} I_{\text{THRU LOOP}}$$

$$\parallel$$

OR

$$\oint_{\text{LOOP}} \bar{B} \cdot d\bar{\ell} = \frac{4\pi I}{c}$$

IN SITUATIONS OF HIGH SYMMETRY AMPERE'S LAW IS OFTEN THE QUICKEST MEANS OF CALCULATING THE MAGNETIC FIELD.

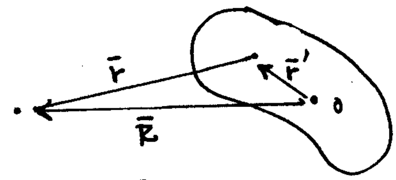
Ph 206 LECTURE 7

MAGNETIC MULTIPOLE EXPANSION (BECKER SEC 47)

WE BEGIN OUR DISCUSSION OF METHODS OF SOLUTION FOR THE MAGNETIC FIELD $\bar{B}$ WITH THE CASE OF AN OBSERVER EXTERNAL TO A LOCALISED CURRENT DISTRIBUTION.

TO MINIMIZE THE COMPLEXITY OF THE INTEGRATION, WE CONSIDER THE VECTOR POTENTIAL $\bar{A}$, AND USE $\bar{B} = \bar{\nabla} \times \bar{A}$, RATHER THAN THE BIOT-SAVART LAW:

$$\bar{A} = \frac{1}{c}\int \frac{\bar{J}'\,dvol'}{r} \quad \text{WITH} \quad \bar{r} = \bar{R} - \bar{r}'$$



IT WILL PROVE MUCH SIMPLER TO ADOPT A MICROSCOPIC VIEWPOINT, IN WHICH THE CURRENT $\bar{J}$ IS THOUGHT TO BE DUE TO VARIOUS CHARGES e_i' WITH VELOCITIES $\bar{v}_i'$.

$$\text{THEN} \quad \int \bar{J}'\,dvol' \longleftrightarrow \sum_i e_i'\bar{v}_i'$$

WE DROP THE SUMMATION INDEX i TO SIMPLIFY THE NOTATION.

$$\bar{A} = \frac{1}{c}\sum \frac{e'\bar{v}'}{r}$$

AS BEFORE WE EXPAND $\displaystyle \frac{1}{r} = \frac{1}{R} + \frac{\hat{R}\cdot\bar{r}'}{R^2} + \frac{3(\hat{R}\cdot\bar{r}')^2 - r'^2}{2R^3} + \cdots$

$$\text{SO} \quad \bar{A} = \frac{1}{cR}\sum e'\bar{v}' + \frac{1}{cR^2}\sum e'\bar{v}'(\hat{R}\cdot\bar{r}') + \cdots$$

THE FIRST TERM IS A PERFECT TIME DERIVATIVE: $\displaystyle \sum e'\bar{v}' = \frac{d}{dt}\sum e'\bar{r}'$

IN <u>MAGNETOSTATICS</u> THE TIME DERIVATIVE OF ANY PROPERTY OF THE CURRENTS MUST VANISH. THUS, THERE IS NO WAY IN WHICH A STEADY CURRENT DISTRIBUTION CAN SIMULATE THE EFFECT OF A SINGLE MAGNETIC POLE. THIS IS CLEARLY A PLUS FOR AMPERE'S CLAIM THAT ALL KNOWN MAGNETIC EFFECTS ARE DUE TO CURRENTS. [MACROSCOPIC VIEW: $\bar{A} = \frac{1}{cR}\int \bar{J}\,dvol' + \cdots$. IF CURRENTS FLOW IN <u>CLOSED LOOPS</u> THEN $\int \bar{J}\,dvol' = \sum_{\text{LOOPS}} I \oint d\bar{l} = 0$.]

THE 2ND TERM IN THE EXPANSION OF $\bar{A}$ IS MORE COMPLEX, BUT IT LOOKS LIKE A PIECE OF A TRIPLE CROSS PRODUCT. AFTER SOME PLAYING, WE FIND WE CAN WRITE

$$\bar{v}'(\hat{R}\cdot\bar{r}') = \frac{1}{2}\frac{d}{dt}\bar{r}'(\hat{R}\cdot\bar{r}') + \frac{1}{2}\left[\bar{v}'(\hat{R}\cdot\bar{r}') - \bar{r}'(\hat{R}\cdot\bar{v}')\right]$$

($\hat{R}$ IS A CONSTANT VECTOR)

of $\hat{R}$ PIECE $\bar{v}'(\hat{R}\cdot\bar{r}')$

AGAIN, THE TIME DERIVATIVE OF THE SUM: $\frac{d}{dt}\sum e'\bar{r}'(\hat{R}\cdot\bar{r}')$ VANISHES IN MAGNETOSTATICS. (THIS SUM IS A PIECE OF THE QUADRUPOLE MOMENT.) THUS THE LOWEST ORDER SURVIVING TERM IN THE EXPANSION OF $\bar{A}$ IS

$$\bar{A} = \frac{1}{2cR^2}\sum(\bar{r}'\times e\bar{v}')\times\hat{R} \quad + \cdots\cdots$$

WE INTRODUCE THE DEFINITION $\bar{m} = \frac{1}{2c}\sum \bar{r}\times e\bar{v}$

= <u>MAGNETIC</u> <u>DIPOLE</u> <u>MOMENT</u>

THEN $\bar{A}_{\text{DIPOLE}} = \frac{\bar{m}\times\hat{R}}{R^2}$ (WE NOW DROP THE ' LABELLING QUANTITIES OF THE CURRENT DISTRI.)

IF WE WISH TO RETURN TO A DESCRIPTION IN TERMS OF CONTINUOUS CURRENT DISTRIBUTIONS, THEN CLEARLY

$$\bar{m} = \frac{1}{2c}\int \bar{r}\times\bar{J}\; d\text{vol}$$

WE JUSTIFY THE NAME 'DIPOLE' BY CALCULATING THE FIELD $\bar{B} = \bar{\nabla}\times\bar{A}$.

WE NOTE THAT $\nabla\times(\bar{a}\times\bar{b}) = \bar{a}(\bar{\nabla}\cdot\bar{b}) - \bar{b}(\bar{\nabla}\cdot\bar{a}) + (\bar{b}\cdot\bar{\nabla})\bar{a} - (\bar{a}\cdot\bar{\nabla})\bar{b}$,

SO $$\bar{\nabla}\times\left(\bar{m}\times\frac{\bar{R}}{R^3}\right) = \bar{m}\left(\bar{\nabla}\cdot\frac{\bar{R}}{R^3}\right) - (\bar{m}\cdot\bar{\nabla})\frac{\bar{R}}{R^3}$$

$$\bar{\nabla}\cdot\frac{\bar{R}}{R^3} = \frac{\bar{\nabla}\cdot\bar{R}}{R^3} - \frac{3(\bar{R}\cdot\bar{\nabla})R}{R^4} = \frac{3}{R^3} - 3\frac{\bar{R}\cdot\bar{R}}{R^5} = 0 \qquad (\text{FOR } R>0)$$

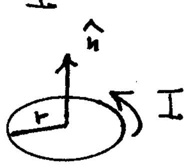
AND $$(\bar{m}\cdot\bar{\nabla})\frac{\bar{R}}{R^3} = \frac{(\bar{m}\cdot\bar{\nabla})\bar{R}}{R^3} - \frac{3\bar{R}}{R^4}(\bar{m}\cdot\bar{\nabla})R = \frac{\bar{m}}{R^3} - \frac{3\bar{R}(\bar{m}\cdot\bar{R})}{R^5}$$

SO $$\bar{B} = \bar{\nabla}\times\bar{A} = \frac{3(\bar{m}\cdot\hat{R})\hat{R} - \bar{m}}{R^3} \qquad (\text{FOR } R>0)$$

THIS IS EXACTLY THE FORM OF THE ELECTRIC DIPOLE FIELD FOUND IN LECTURE 1.

<u>EXAMPLE</u> A CURRENT LOOP OF RADIUS r, CURRENT I

THEN $\bar{m} = \frac{1}{2c}\int \bar{r}\times I d\bar{l} = \frac{1}{2c}2\pi r^2 I\,\hat{n} = \frac{IA}{c}\hat{n}$



WHERE $A = \pi r^2$ = AREA OF THE LOOP.

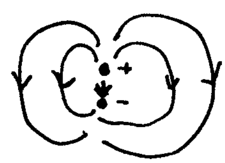
THUS WE ARE LED TO THE NOTION THAT PERMANENT MAGNETS WHICH EXHIBIT MAGNETIC-DIPOLE FIELD PATTERNS ARE MADE OUT OF 'MOLECULAR CURRENT LOOPS'. IT TURNS OUT THAT A DETAILED DESCRIPTION OF THESE LOOPS TAKES US OUT OF THE REALM OF CLASSICAL PHYSICS AND INTO THE QUANTUM ERA. WHILE ELECTRONS IN A CIRCULAR ORBIT ABOUT A NUCLEUS CAN EASILY BE IMAGINED TO CAUSE A CURRENT LOOP, IT IS ALSO FOUND THAT AN ISOLATED ELECTRON HAS A MAGNETIC MOMENT OF SIZE $\frac{e\hbar}{2mc}$. BUT EXPERIMENTALLY IT HAS NOT BEEN POSSIBLE TO ASSIGN A MEANINGFUL RADIUS TO THE 'CIRCULATING CURRENT' OF AN ELECTRON'S CHARGE.

THIS BRINGS UP THE QUESTION OF WHETHER ELECTRONS (OR PROTONS AND NEUTRONS WHICH ALSO POSSESS AN INTRINSIC MAGNETIC MOMENT) MIGHT ACTUALLY CONTAIN TWO OPPOSITE MAGNETIC POLES LEADING TO A DIPOLE - RATHER THAN A CIRCULATING CURRENT A LA AMPERE.

A DISTINCTION BETWEEN THESE TWO CASES CAN BE MADE BY EXAMINING THE MAGNETIC FIELD DUE TO THE INTRINSIC MOMENT IN THE LIMIT $R \to 0$.

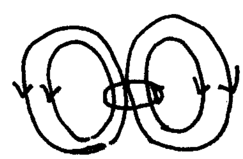
IF THE DIPOLE MOMENT IS DUE TO TWO OPPOSITE POLES SLIGHTLY SEPARATED, THEN

$$\bar{B} = \frac{3(\bar{m}\cdot\hat{R})\hat{R} - \bar{m}}{R^3} - \frac{4\pi\bar{m}}{3}\,\delta(\bar{R})$$



EXACTLY AS CONSIDERED ON PROBLEM SET 1 FOR ELECTRIC DIPOLES.

FOR A MAGNETIC DIPOLE DUE TO A CURRENT LOOP, WE EXPECT A DELTA FUNCTION TERM OF THE OPPOSITE SIGN.



IN DETAIL, WE CONSIDER $\int \bar{B}\,d\text{vol}$ FOR A SMALL SPHERE OF RADIUS R ABOUT THE DIPOLE $\bar{m}$.

$$\int \bar{B}\,d\text{vol} = \int \bar{\nabla}\times\bar{A}\,d\text{vol} = \int_{\text{SURFACE}} d\bar{S}\times\bar{A} = R^2\int_{\text{SURFACE}} d\Omega\,\hat{R}\times\bar{A}$$

USING YET ANOTHER VARIATION ON GAUSS' LAW.

NOW $\bar{A} = \frac{\bar{m}\times\hat{R}}{R^2}$ SO $\int \bar{B}\,d\text{vol} = \int_{\text{SURFACE}} d\Omega\,(\bar{m} - (\bar{m}\cdot\hat{R})\hat{R}) = \frac{8\pi\bar{m}}{3}$

HENCE $$\bar{B} = \frac{3(\bar{m}\cdot\hat{R})\hat{R} - \bar{m}}{R^3} + \frac{8\pi\bar{m}}{3}\,\delta(\bar{R}) \qquad \text{FOR CURRENTS.}$$

CERTAIN EXPERIMENTS IN ATOMIC SPECTROSCOPY ACTUALLY PROBE THE MAGNETIC FIELD STRENGTH AT $\bar{R}=0$ (THE SO-CALLED HYPERFINE INTERACTION), AND GIVE EVIDENCE IN FAVOR OF THE CURRENT-LOOP ORIGIN OF THE ELECTRON'S MAGNETIC MOMENT. THUS FAR AMPERE'S CLAIM IS STILL VALID!

Forces and Torques on a Magnetic Dipole

SUPPOSE A MAGNETIC DIPOLE OF CONSTANT MAGNITUDE M IS SUBJECT TO AN <u>EXTERNAL</u> MAGNETIC FIELD. WHAT FORCES AND TORQUES DOES IT FEEL?

OUR FORCE LAW FROM p 80 IS $\bar{F} = \frac{1}{c}\int \bar{J}\times\bar{B}\,dvol$

WE CAN EXPAND THE EXTERNAL FIELD ABOUT THE CENTER OF THE DIPOLE

$$\bar{B}(\bar{r}) = \bar{B}(0) + (\bar{r}\cdot\bar{\nabla})\bar{B}(0) + \cdots$$

$$\text{SO}\quad \bar{F} = -\frac{1}{c}\bar{B}(0)\times\int\bar{J}\,dvol + \frac{1}{c}\int\bar{J}\times(\bar{r}\cdot\bar{\nabla})\bar{B}(0)\,dvol + \cdots$$

THE FIRST INTEGRAL VANISHES FOR A STEADY CURRENT DISTRIBUTION AS BEFORE. THE SECOND INTEGRAL CAN BE MANIPULATED WITH CONSIDERABLE EFFORT INTO THE FORM

$$\underline{\bar{F}} = \bar{\nabla}\,\underline{(\bar{m}\cdot\bar{B}(0))} = \underline{(\bar{m}\cdot\bar{\nabla})\,\bar{B}(0)} \qquad \left(\text{USING } \nabla\times B\big|_0 = 0 \text{ FOR THE EXTERNAL FIELD}\right)$$

A DERIVATION: SWITCH FROM $\int\bar{J}\ldots\,dvol$ TO $\sum e\bar{v}'\ldots$

$$\bar{F} = \frac{1}{c}\sum e\,\bar{v}'\times(\bar{r}'\cdot\nabla)\,\bar{B}(0) + \cdots$$

LOOK AT JUST THE x COMPONENT: $F_x = \frac{1}{c}\sum e\,v_y'\,(\bar{r}'\cdot\nabla)B_z(0) - \frac{1}{c}\sum e\,v_z'(\bar{r}'\cdot\nabla)B_y(0)$

$$\text{NOW}\quad \frac{1}{c}\sum e\,v_y'\,(\bar{r}'\cdot\bar{\nabla})B_z(0) = \frac{1}{c}\sum e\,v_y'\,(\bar{r}'\cdot\underbrace{\bar{\nabla}B_z(0)}_{\text{DOES NOT DEPEND ON }\bar{r}'})$$

$$= \frac{1}{2c}\sum e\underbrace{\frac{d}{dt}\,r_y'\,(\bar{r}'\cdot\bar{\nabla}B_z(0))}_{0 \text{ IN STATICS}} + \frac{1}{2c}\sum e\left[v_y'\,(\bar{r}'\cdot\bar{\nabla}B_z(0)) - r_y'\,(\bar{v}'\cdot\bar{\nabla}B_z(0))\right]$$

$$= \frac{1}{2c}\sum e\left[(\bar{r}'\times\bar{v}')\times\bar{\nabla}B_z(0)\right]_y = \left[\bar{m}\times\bar{\nabla}B_z(0)\right]_y = m_z\frac{\partial B_z(0)}{\partial x} - m_x\frac{\partial B_z(0)}{\partial z}$$

LIKE WISE $\frac{1}{c}\sum' e\, v_z'(\bar{r}'\cdot\bar{\nabla})B_y(0) = m_x \frac{\partial B_y(0)}{\partial y} - m_y \frac{\partial B_y(0)}{\partial x}$

THUS, $F_x = -m_x\left(\frac{\partial B_y(0)}{\partial y} + \frac{\partial B_z(0)}{\partial z}\right) + m_y \frac{\partial B_y(0)}{\partial x} + m_z \frac{\partial B_z(0)}{\partial x}$

$$\bar{\nabla}\cdot\bar{B} = 0 \Rightarrow \frac{\partial B_y(0)}{\partial y} + \frac{\partial B_z(0)}{\partial z} = -\frac{\partial B_x(0)}{\partial x}$$

SO, $F_x = m_x \frac{\partial B_x(0)}{\partial x} + m_y \frac{\partial B_y(0)}{\partial x} + m_z \frac{\partial B_z(0)}{\partial x}$

$= \frac{\partial}{\partial x}(\bar{m}\cdot\bar{B}(0))$ SINCE $\bar{m}$ IS CONSTANT IN THIS DERIVATION

$\Rightarrow \bar{F} = \bar{\nabla}(\bar{m}\cdot\bar{B}(0)) = (\bar{m}\cdot\bar{\nabla})\bar{B}(0)$

USING $\bar{\nabla}\times\bar{B}(0) = 0$ FOR THE EXTERNAL FIELD

IF WE SET U = ENERGY STORED IN CONFIGURATION OF THE DIPOLE

THEN $\bar{F} = -\bar{\nabla}U$ SO $\underline{U = -\bar{m}\cdot\bar{B}(0)}$

WE CAN VERIFY THE VALIDITY OF THIS LAST ARGUMENT BY CONSIDERING THE TORQUE:

$$\bar{N} = \frac{1}{c}\int \bar{r}\times(\bar{J}\times\bar{B})\,d\mathrm{vol} \approx \frac{1}{c}\int \bar{r}\times(\bar{J}\times B(0))\,d\mathrm{vol} + \ldots$$

WE CAN MANIPULATE THIS MORE EASILY BY REVERTING TO A DESCRIPTION INVOLVING INDIVIDUAL CHARGES:

$$\bar{N} = \sum \frac{e}{c}\left(\bar{r} \times (\bar{v} \times \bar{B}(0)\right) = \sum \frac{e}{c}\left((\bar{r}\cdot\bar{B}(0))\,\bar{v} - (\bar{r}\cdot\bar{v})\,\bar{B}(0)\right).$$

NOW $\sum e(\bar{r}\cdot\bar{v}) = \frac{1}{2}\frac{d}{dt}\sum e r^2 = 0$ FOR STEADY CURRENTS.

LIKEWISE $\sum \frac{e}{c}(\bar{r}\cdot\bar{B}(0))\,\bar{v} = \frac{1}{2}\frac{d}{dt}\sum \frac{e}{c}\bar{r}\,(\bar{r}\cdot\bar{B}(0)) + \frac{1}{2c}\sum(\bar{r}\times e\bar{v}) \times \bar{B}(0)$

SO $\underline{\bar{N} = \bar{m} \times \bar{B}(0)} + \ldots$.

THIS IS CONSISTENT WITH OUR ENERGY RELATION $U = -\bar{m}\cdot\bar{B}(0)$, AS WE FOUND WHEN CONSIDERING ELECTRIC DIPOLES.

EXAMPLE LARMOR PRECESSION

SUPPOSE WE SOMEHOW HAVE A PARTICLE OF CHARGE e, MASS m IN A CIRCULAR ORBIT OF RADIUS r, MOVING WITH UNIFORM VELOCITY v.

THEN THE MAGNETIC MOMENT IS $\bar{m} = \frac{1}{2c}\bar{r}\times e\bar{v} = \frac{erv}{2c}\hat{n}$

WHERE $\hat{n}$ = NORMAL TO THE ORBIT.

BUT THE ANGULAR MOMENTUM ABOUT THE CENTER IS $\bar{L} = \bar{r}\times m\bar{v} = mrv\,\hat{n}$

SO $\bar{m} = \frac{e}{2mc}\bar{L} \equiv \Gamma\bar{L}$

(AN ELECTRON HAS MAGNETIC MOMENT $\frac{e\hbar}{2mc}$ AND INTRINSIC ANGULAR MOMENTUM (SPIN) OF $\hbar/2$. THUS THESE TWO QUANTITIES ARE RELATED IN A NON-CLASSICAL WAY)

IF OUR ORBITING CHARGE IS SUBJECTED TO A MAGNETIC FIELD $\bar{B}$,

THEN $\bar{N} = \bar{m}\times\bar{B} = \frac{d\bar{L}}{dt}$, SO $\frac{d\bar{L}}{dt} = \Gamma\bar{L}\times\bar{B} = -\Gamma\bar{B}\times\bar{L}$.

WE RECOGNIZE THIS AS IMPLYING STEADY PRECESSION OF $\bar{L}$ (AND $\bar{m}$) ABOUT $\bar{B}$ AT RATE $\bar{\Omega} = -\Gamma\bar{B}$.

Ph 206 LECTURE 7

IN 1915 EINSTEIN AND DE HAAS CARRIED OUT AN EXPERIMENT DESIGNED TO LOOK FOR THE ANGULAR MOMENTUM ASSOCIATED WITH THE MAGNETISATION. THEY FELT THIS WOULD BE CLEAR EVIDENCE FOR THE EXISTENCE OF THE 'ATOMIC CURRENTS' OF AMPERE - AS OPPOSED TO MAGNETIC MONOPOLES.

THEY HUNG A MAGNETISED IRON CYLINDER FROM A THIN WIRE - AND THEN DEMAGNETISED THE IRON WITH AN ELECTROMAGNET NOT IN PHYSICAL CONTACT WITH THE IRON CYLINDER. WHEN THE MAGNETISATION HAS VANISHED, THE ANGULAR MOMENTUM OF THE ROD WILL HAVE CHANGED BY

$$\Delta \bar{L} = \frac{2mc}{e} \bar{M}_0$$

AND THE CYLINDER WILL BE ROTATING!

EINSTEIN AND DE HAAS CLAIMED TO HAVE VERIFIED THE ABOVE RELATION TO WITHIN 10%. HOWEVER, OTHER EXPERIMENTS IN THE FOLLOWING YEARS (INCLUDING ONE AT PRINCETON) SHOWED

$$\Delta \bar{L} = \frac{mc}{e} \bar{M}_0$$

EVENTUALLY THIS WAS INTERPRETED AS EVIDENCE THAT THE ELECTRON HAS INTRINSIC ANGULAR MOMENTUM AND MAGNETIC MOMENT RELATED BY

$$m = \frac{e}{mc} L = \frac{e\hbar}{2mc}$$

OF COURSE, IF WE ALLOW AN ELECTRON TO HAVE INTRINSIC ANGULAR MOMENTUM, PERHAPS A MAGNETIC MONOPOLE COULD HAVE SOME ALSO....

Historical Note on the Concept of Current

[See also the book: J.Z. Buchwald, 'From Maxwell to Microphysics']

Just as Maxwell's view of charge (p. 23a; sec. III, Vol I of Treatise) was different from ours, so was his view of electrical currents.

We think of current as due to the motion of point charges. The current density can be written $\bar{J} = \rho_+\bar{v}_+ + \rho_-\bar{v}_-$ (for + & − charges with different velocities). We interpret the observed relation $\bar{J} = \sigma\bar{E}$ as due to inelastic collisions of electrons, resulting in loss of energy from the electromagnetic field at a rate $\rho\bar{v}\cdot E = E^2/\sigma$

In Maxwell's view 'charge' is a reinterpretation of electric displacement that has nonzero divergence: $\rho = \bar{\nabla}\cdot\bar{D}/4\pi$.

So if the displacement $\bar{D}$ were time dependent it appeared natural to Maxwell to associate this with a 'displacement current', $(1/4\pi)\partial\bar{D}/\partial t$. This current, however, dissipates no energy.

More problematic in Maxwell's view was the 'conduction' current $\bar{J}$ that dissipated energy. Maxwell seems to have developed a pre-Drude model (sec. 111) in which electric displacement was not steady on the microscale – but rose rapidly until some kind of 'breakdown' occured in which the field energy $\epsilon E^2/8\pi$ was dissipated. He considered the ratio ϵ/σ to set the time scale for this process, and noted that this is quite short in metals.

Another obscure point in Maxwell's view was how electromagnetism produces mechanical force on conductors. He wrote (the equivalent of) $\bar{F} = \bar{J}/c \times \bar{B}$ for the force on a current carrying conductor (sec. 603) but argued that the current itself experienced no force (sec. 501).

The Hall effect (1880) was the first evidence that is view was wrong.

Heaviside (1889) was the first to write $\bar{F} = q(\bar{E} + \frac{\bar{v}}{c}\times\bar{B})$ for the force on a moving charge – but as a 'Maxwellian' he was very uncomfortable with the idea that all charge resides on small particles in contrast to the notion that charge is a manifestation of strain on the ether.

It was left to H.A. Lorentz in the late 1890's to develop the 'modern' view in which electricity is based on electrons.

air, or passes through a magnet, or soft iron, or any other substance, whether paramagnetic or diamagnetic.

500.] When a circuit is placed in a magnetic field the mutual action between the current and the other constituents of the field depends on the surface-integral of the magnetic induction through any surface bounded by that circuit. If by any given motion of the circuit, or of part of it, this surface-integral can be *increased*, there will be a mechanical force tending to move the conductor or the portion of the conductor in the given manner.

The kind of motion of the conductor which increases the surface-integral is motion of the conductor perpendicular to the direction of the current and across the lines of induction.

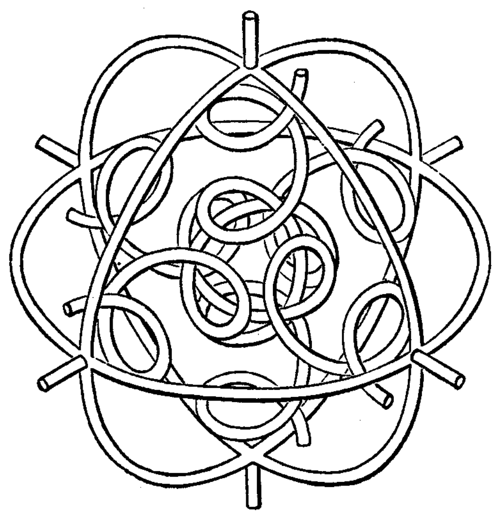


Fig. 25.

Relations between the positive directions of motion and of rotation indicated by three right-handed screws.

If a parallelogram be drawn, whose sides are parallel and proportional to the strength of the current at any point, and to the magnetic induction at the same point, then the force on unit of length of the conductor is numerically equal to the area of this parallelogram, and is perpendicular to its plane, and acts in the direction in which the motion of turning the handle of a right-handed screw from the direction of the current to the direction of the magnetic induction would cause the screw to move.

Hence we have a new electromagnetic definition of a line of

magnetic induction. It is that line to which the force on the conductor is always perpendicular.

It may also be defined as a line along which, if an electric current be transmitted, the conductor carrying it will experience no force.

501.] It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it. If the conductor be a rotating disk or a fluid it will move in obedience to this force, and this motion may or may not be accompanied by a change of position of the electric current which it carries. [But if the current itself be free to choose any path through a fixed solid conductor or a network of wires, then, when a constant magnetic force is made to act on the system, the path of the current through the conductors is not permanently altered, but after certain transient phenomena, called induction currents, have subsided, the distribution of the current will be found to be the same as if no magnetic force were in action.]*

The only force which acts on electric currents is electromotive force, which must be distinguished from the mechanical force which is the subject of this chapter.

* {Mr. Hall has discovered (*Phil. Mag.* ix. p. 225, x. p. 301, 1880) that a steady magnetic field does slightly alter the distribution of currents in most conductors, so that the statement in brackets must be regarded as only approximately true.}

FROM VOL II. OF MAXWELL'S TREATISE

Ph 206 LECTURE 7

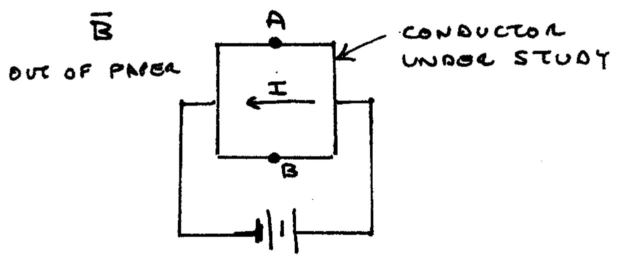
THE HALL EFFECT

IN 1879 E. HALL WAS A GRADUATE STUDENT AT JOHNS HOPKINS UNDER ROWLAND, AND HE WAS PUZZLED BY MAXWELL'S SEC. 501 IN WHICH IT IS CLAIMED THAT ELECTRIC CURRENTS ARE UNAFFECTED BY MAGNETIC FIELDS – ONLY THE CONDUCTOR THAT CARRIES THE CURRENT WAS THOUGHT TO EXPERIENCE THE $\frac{\vec{J}}{c}\times\vec{B}$ FORCE.

HALL'S NOTEBOOK, NOV. 1879:

If electricity were an incompressible fluid it might be acted on in a particular direction without moving in that direction. I took an example like this. Suppose a stream of water flowing in a perfectly smooth pipe which is however loosely fitted with gravel. The water will meet with resistance from the gravel but none from the pipe at least no frictional resistance. Suppose now some body brought near the pipe which has the power of attracting a *stream* of water. The water would evidently be pressed against the side of the pipe but being incompressible and, with the gravel, completely filling the pipe, it could not move in the direction of the pressure and the result would simply be a state of stress without any actual change of course by the stream. ["It is evident however that in such a case the pipe might be tapped on the side toward the attracting object and a second pipe applied to the orifice" crossed out.] It is evident however that if a hole were made transversely through the pipe in the direction of the pressure and the two orifices thus made were connected by a second pipe, water would flow *out toward* the attracting object and *in* at the opposite orifice. This supposes of course that the attracting object acts upon the current flowing in one direction without acting, equally at least, upon the current in the other direction.

Nov. 4th, '79. I mean by this that the attracting object is supposed to act, not upon the water at rest and under all circumstance, but only when the water is flowing and flowing in a certain direction or the opposite. In this way I arrived at the conclusion that in order to show conclusively that the magnet does not affect the current at all I must show not merely that there was no actual deflection of the current which seemed to be already shown by my experiments on resistance, but further that there was no tendency of the current to move. In order to do this I tried to repeat an experiment which Prof. Rowland had once tried without any positive result.[1]



HALL'S EXPERIMENT: ('MODERN' DESCRIPTION)

CURRENT FLOWS IN A CONDUCTOR THAT IS IN A MAGNETIC FIELD $B \perp$ TO I. THEN CHARGE q MOVING WITH VELOCITY v FEELS FORCE $q\frac{v}{c}B$ ALONG LINE AB.

CHARGES REARRANGE IN THE CONDUCTOR UNTIL A FIELD E_H IS CREATED ALONG THE AB DIRECTION TO CANCEL THE $\vec{v}\times\vec{B}$ FORCE:

$$\bar{F} = 0 = qE_H + \frac{qv}{c}B. \quad \text{THUS} \quad E_H = -\frac{v}{c}B = (V_A - V_B)/\text{LENGTH AB}.$$

1. IF THE MOVING CHARGES WERE POSITIVE, + CHARGE WOULD ACCUMULATE ALONG EDGE A AND $V_A - V_B > 0$. BUT IF THE CHARGES WERE NEGATIVE, – CHARGE WOULD ACCUMULATE AT A AND $V_A - V_B < 0$. THE EXPERIMENT SHOWED THAT $V_A - V_B < 0$ FOR METALS

   $\Longrightarrow$ THE CHARGE THAT MOVES IN METALS IS NEGATIVE!

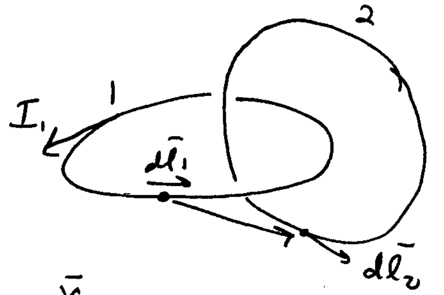
2. HALL REARRANGED HIS FORMULA SLIGHTLY AND REPORTED THE QUANTITY R_H DEFINED BY $E_H = -R_H J B = -R_H nqv B$, NOTING THAT CURRENT DENSITY J IS nqv WHERE n = # OF CHARGES / VOLUME. COMPARING WITH ABOVE WE CAN INTERPRET $R_H = \frac{1}{nqc}$. SO IF q IS KNOWN, THEN n CAN BE DETERMINED.

BUT $q = q_e$ WAS NOT KNOWN IN HALL'S TIME! INSTEAD, USE $J = \sigma E_0$ WHERE E_0 IS THE FIELD CREATED BY THE BATTERY IN THE CONDUCTOR. THUS $E_H = -R_H\sigma E_0 B = -\frac{v}{c}B$ SO $v = cR_H\sigma E_0$. FOR COPPER, $R_H = 6\times10^{-25}$ cgs UNITS, $\sigma = 9\times10^{17}$ cgs

THEN FOR $E_0 = \frac{1\,\text{VOLT}}{\text{cm}} = \frac{1}{300}$ cgs, $v = 50\,\text{cm/sec}$. THIS WAS THE FIRST INDICATION OF THE SPEED OF CHARGE IN ELECTRICAL CURRENTS

Ph 206 Lecture 7

Ampere, Biot-Savart and Knots



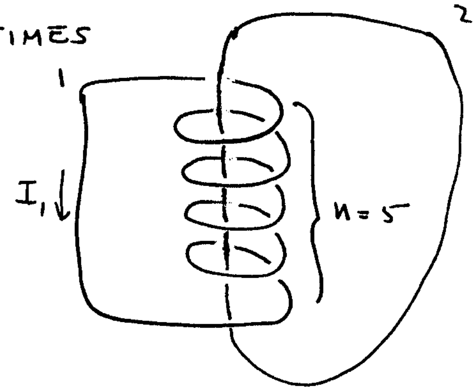
AMPERE: $\oint_2 \bar{B}_2 \cdot d\bar{l}_2 = \frac{4\pi}{c} I_1$

BIOT-SAVART: $\bar{B}_2 = \frac{I_1}{c} \oint_1 \frac{d\bar{l}_1 \times \bar{r}}{r^3}$, $\bar{r} \equiv \bar{r}_2 - \bar{r}_1$

COMBINING, WE OBTAIN THE IDENTITY:

$$1 = \frac{1}{4\pi} \oint_1 \oint_2 \frac{d\bar{l}_1 \times \bar{r} \cdot d\bar{l}_2}{r^3} = -\frac{1}{4\pi} \oint_1 \oint_2 \frac{\bar{r} \cdot d\bar{l}_1 \times d\bar{l}_2}{r^3} = \frac{1}{4\pi} \oint_1 \oint_2 \frac{(\bar{r}_1 - \bar{r}_2) \cdot d\bar{l}_1 \times d\bar{l}_2}{|\bar{r}_1 - \bar{r}_2|^3}$$

NOW SUPPOSE CIRCUIT 1 LINKS LOOP 2 n TIMES



AMPERE NOW SAYS $\oint \bar{B}_2 \cdot d\bar{l}_2 = \frac{4\pi n}{c} I_1$

SO THE IDENTITY BECOMES

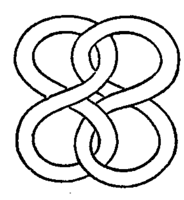
$$n = \frac{1}{4\pi} \oint_1 \oint_2 \frac{(\bar{r}_1 - \bar{r}_2) \cdot d\bar{l}_1 \times d\bar{l}_2}{|\bar{r}_1 - \bar{r}_2|^3} .$$

THAT IS, WE CAN CALCULATE THE NUMBER OF LINKS OF TWO LOOPS (= A KNOT) BY A GEOMETRIC INTEGRAL OVER THE LOOPS. THE NUMBER n IS INDEPENDENT OF THE SHAPE OF THE LOOPS!

THIS RESULT IS DUE TO GAUSS (1833), AND IS THE FOUNDATION OF KNOT THEORY. SEE, MAXWELL, SECS. 417-422 (P. 39, VOL. II, DOVER REPRINT EDITION)

ALSO SEE, A.C. HIRSHFELD, AM. J. PHYS. <u>66</u>, 1060 (1998).

KNOTS ARE A TWISTY BUSINESS: THE LOOPS SHOWN → ARE INTERTWINED, BUT THE LINKING NUMBER n OF GAUSS IS ZERO...



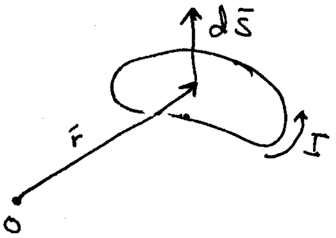
(FROM MAXWELL)

Ph 206 Lecture 7

Solid Angle, Magnetic Scalar Potential and Knots

Gauss also thought of another interesting view of knots & magnetism.

This relates to solid angle: $\Omega = \int_S \frac{\hat{r}\cdot d\bar{s}}{r^2} = \int_S \frac{\bar{r}\cdot d\bar{s}}{r^3}$,



where $\bar{r}$ points from the observer to the surface.

Suppose the surface S is bounded by a current loop

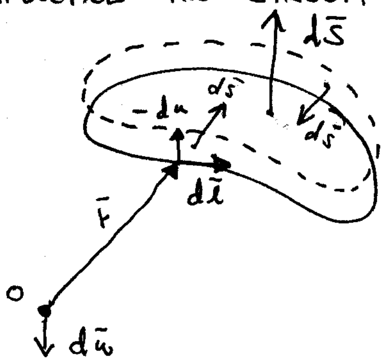
Then Biot-Savart say $\bar{B}(\text{at } O) = -\frac{I}{c}\oint \frac{d\bar{l}\times\bar{r}}{r^3}$; note the sign. The usual convention in Biot-Savart is that $\bar{r}$ points from current to observer.

This can be related to solid angle by an amusing trick.

Suppose the observer is displaced by a small amount $d\bar{u}$. The solid angle subtended by the circuit then changes by

$$d\Omega = \bar{\nabla}\Omega \cdot d\bar{u}.$$

But, this change is equivalent to that due to displacing the circuit by $-d\bar{u}$ and leaving the observer fixed.



In this case, the change in solid angle is just that of the band of height $-d\bar{u}$ that runs around the loop

$$\delta\bar{S} = (-d\bar{u})\times d\bar{l} = d\bar{l}\times d\bar{u}$$

$$\delta\Omega = \frac{\bar{r}\cdot d\bar{l}\times d\bar{u}}{r^3} = -d\bar{u}\cdot\frac{d\bar{l}\times\bar{r}}{r^3}$$

$$d\Omega = \int \delta\Omega = -d\bar{u}\cdot\oint\frac{d\bar{l}\times\bar{r}}{r^3} = \bar{\nabla}\Omega\cdot d\bar{u}$$

$$\Rightarrow \quad \bar{\nabla}\Omega = -\int\frac{d\bar{l}\times\bar{r}}{r^3} = \frac{c}{I}\bar{B}$$

so $\bar{B} = -\bar{\nabla}\Phi_M$ where $\Phi_M = \frac{-I}{c}\Omega$ (noting the sign of $\bar{r}$)

Φ_M is called the magnetic scalar potential. It only applies outside current distributions, where $\bar{\nabla}\times\bar{B} = 0$.

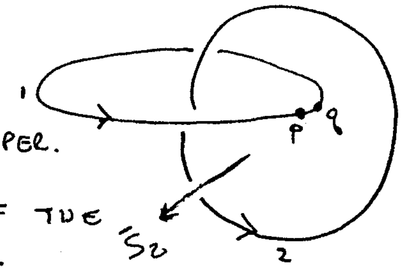
Applying this to the knot problem:

$$\frac{4\pi n I}{c} = \oint \bar{B}\cdot d\bar{l} = +\frac{I}{c}\oint\bar{\nabla}\Omega\cdot d\bar{l} \quad \text{so} \quad n = \frac{1}{4\pi}\oint\bar{\nabla}\Omega\cdot d\bar{l}$$

Now comes another tricky point. If the solid angle were an 'ordinary' well-behaved function, we would have $\oint \bar{\nabla}\Omega \cdot d\bar{l} = 0$.

But this is not the case when knots are involved!

For example, consider point p, just this side of the plane of loop 2.



Then $\Omega(p) \approx -2\pi$, defining $d\bar{S}_2$ as out of paper.

But, at point q, just on the other side of the plane of loop 2, we have $\Omega(q) \approx +2\pi$.

That is, Ω takes a step of 4π as we pass thru loop 2.

$$\oint_1 \bar{\nabla}\Omega \cdot d\bar{l} = 4\pi \quad \text{if loop 1 passes once thru loop 2}$$

$$\oint \bar{\nabla}\Omega \cdot d\bar{l} = 4\pi n \quad \text{- - - - - -} \quad n \text{ times} \quad \text{- - - -}$$

So the result on p 88e, $n = \frac{1}{4\pi}\oint \bar{\nabla}\Omega \cdot d\bar{l}$ is consistent!

The linking number n is, however, not sufficient for a full characterization of knots. For example, the two cases

Both have linking number $n = 0$, but the left case is not knotted, while the right case is...