SOURCES OF THE MAGNETIC FIELD (BECKER SEC. 38 & 47)

BEFORE CONTINUING OUR STUDY OF MAGNETOSTATICS, WE MAKE A DIGRESSION IN THE SPIRIT OF MAXWELL, WHICH GIVES A FIRST LOOK INTO NON-STATIC ELECTRO-MAGNETIC SITUATIONS.

WE EXPECT WE SHALL BE ABLE TO RECONSTRUCT THE MAGNETIC FIELD BY STARTING FROM THE DIFFERENTIAL EQUATIONS:

$$\bar{\nabla}\cdot\bar{B} = 0, \qquad \bar{\nabla}\times\bar{B} = \frac{4\pi}{c}\bar{J}, \qquad \bar{\nabla}\cdot\bar{J}_{STATIC} = 0.$$

THUS FAR WE HAVE CONSIDERED THE CURRENT DUE TO THE MOTION OF FREE CHARGES AS THE SOURCE OF THE MAGNETIC FIELD (ALTHOUGH WE HAVE SUGGESTED THAT ELECTRONS AND PROTONS POSSESS INTRINSIC MAGNETIC MOMENTS NOT READILY ASCRIBED TO CLASSICAL MOTION OF FREE CHARGES).

MAXWELL NOTED THAT OTHER PHENOMENA GIVE RISE TO CURRENTS WHICH CAN PRODUCE MAGNETIC FIELDS. HE ALSO HAD THE PROFOUND INSIGHT THAT THE TOTAL "CURRENT" SHOULD ALWAYS FLOW IN CLOSED LOOPS – AS IS THE CASE FOR THE CURRENT OF FREE CHARGES IN MAGNETOSTATICS. SO AS WE CATALOG THE TYPES OF CURRENTS, WE WISH TO END UP WITH THE ADDITIONAL RESULT THAT

$\bar{\nabla}\times\bar{B} = \frac{4\pi}{c}\bar{J}_{TOTAL}$ & SINCE $\bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = 0 \Rightarrow \underline{\bar{\nabla}\cdot\bar{J}_{TOTAL} = 0}$ ($\Rightarrow$ NO SOURCES OR SINKS OF 'CURRENT')

1. FREE CURRENT. THIS IS THE CURRENT, DISCUSSED ABOVE, DUE TO THE MOTION OF CHARGES NOT BOUND TO ATOMS OR MOLECULES.

   FREE CHARGE CONSERVATION TELLS US $\bar{\nabla}\cdot\bar{J}_{FREE} + \frac{\partial\rho_{FREE}}{\partial t} = 0$

2. POLARIZATION CURRENT. A DIELECTRIC MATERIAL POSSESSING POLARIZATION DENSITY $\bar{P}$ CAN BE THOUGHT OF AS ALSO POSSESSING A CHARGE DENSITY

$$\rho_{BOUND} = -\bar{\nabla}\cdot\bar{P} \qquad \text{(LECTURE 2)}$$

IN ELECTROSTATICS, WE NOTED THAT $\rho_{BOUND} = 0$ ALWAYS, ALTHOUGH A NON-TRIVIAL SURFACE CHARGE DENSITY $\sigma = \bar{P}\cdot d\bar{S}$ CAN OCCUR AT A DIELECTRIC BOUNDARY.

IN A NON-ELECTROSTATIC SITUATION, THE POSSIBLE TIME DEPENDENCE OF THE POLARIZATION WOULD LEAD TO A TIME VARIATION OF ρ_{BOUND}, AND HENCE TO A NEW CURRENT $\bar{J}_{BOUND}$.

OF COURSE, WE EXPECT CHARGE CONSERVATION WILL CONTINUE

TO HOLD. i.e. $\bar{\nabla}\cdot\bar{J}_{BOUND} = -\dfrac{\partial \rho_{BOUND}}{\partial t} = \bar{\nabla}\cdot\dfrac{\partial \bar{P}}{\partial t}$

HENCE WE IDENTIFY $\bar{J}_{BOUND} = \dfrac{\partial \bar{P}}{\partial t}$

AS THE POLARISATION CURRENT DENSITY

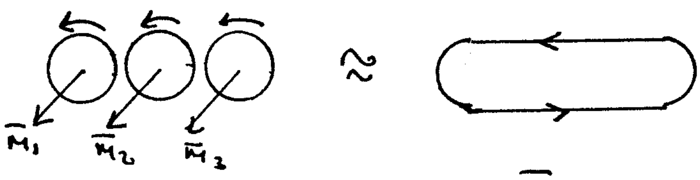
3. <u>MAGNETIZATION CURRENT</u> AS WE WISH TO CONSIDER MORE BELOW, THERE EXIST MATERIALS WITH A NET MAGNETIC DIPOLE DENSITY

$\bar{M} = N\bar{m}$ WHERE $\bar{m}$ = ATOMIC, OR INTRINSIC DIPOLE MOMENT

AND N = NO OF SUCH MOMENTS PER UNIT VOLUME.

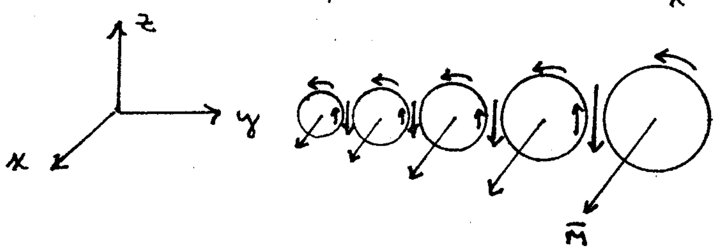
$\bar{M}$ IS CALLED THE <u>MAGNETISATION DENSITY</u>

SPATIAL VARIATIONS IN $\bar{M}$ HAVE THE EFFECT OF A NET CURRENT FLOW. RECALL THE RESULT OF AMPERE THAT A MAGNETIC DIPOLE CAN BE THOUGHT OF AS A CURRENT LOOP:



IF THE MAGNETISATION DENSITY $\bar{M}$ IS UNIFORM, THERE IS NO NET CURRENT FLOW INSIDE THE MATERIAL, ALTHOUGH THERE IS A NET EFFECT AT THE SURFACE, AS SKETCHED ABOVE.

BUT SUPPOSE, FOR EXAMPLE M_x VARIES WITH y



BETWEEN NEIGHBORING LOOPS WE FIND A NET CURRENT IN THE $-z$ DIRECTION, AND $J_z \approx -\dfrac{\partial M_x}{\partial y}$

WHICH SUGGESTS THE RESULT $\bar{J}_{MAGNET.} \approx \bar{\nabla}\times\bar{M}$

THIS CAN BE JUSTIFIED MORE FORMALLY BY CONSIDERING THE VECTOR POTENTIAL

$$\bar{A}_{DIPOLE} = \frac{\bar{m}\times\bar{R}}{R^3}$$

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FOR A MATERIAL CONTAINING MAGNETISATION DENSITY $\bar{M}$, THIS BECOMES

$$\bar{A} = \int \bar{M} \times \frac{\bar{R}}{R^3}\, dvol = \int \bar{M} \times \bar{\nabla}\left(\frac{1}{R}\right) dvol = \int \frac{\bar{\nabla} \times \bar{M}}{R}\, dvol - \int \bar{\nabla} \times \left(\frac{\bar{M}}{R}\right) dvol$$

THE LAST INTEGRAL VANISHES FOR WELL-BEHAVED MATERIALS, BY A VARIATION OF GAUSS' THEOREM:

$$\int \bar{\nabla} \times \frac{\bar{M}}{R}\, dvol = \int_{\text{SURFACE}} d\bar{S} \times \frac{\bar{M}}{R} \to 0$$

FOR A BIG SURFACE SUCH THAT $\bar{M}$ IS ENTIRELY WITHIN.

THEN, SINCE $\bar{A} = \frac{1}{c}\int \frac{\bar{J}}{R}\, dvol$ IS OUR GENERAL SOLUTION,

WE SEE THAT $\underline{\bar{J}_{\text{MAGNETIZATION}} = c\,\bar{\nabla} \times \bar{M}}$.

SINCE $\bar{\nabla}\cdot(\bar{\nabla} \times \bar{M}) = 0$, $\bar{\nabla}\cdot \bar{J}_{\text{MAG}} = 0$ AT ONCE.

4. <u>DISPLACEMENT</u> <u>CURRENT</u> MAXWELL WAS UNSATISFIED WITH THE ABOVE LIST. HE NOTED

$$\bar{\nabla}\cdot\left(\bar{J}_{\text{FREE}} + \bar{J}_{\text{BOUND}} + \bar{J}_{\text{MAG}}\right) = -\frac{\partial \rho_{\text{FREE}}}{\partial t} - \frac{\partial \rho_{\text{BOUND}}}{\partial t} = -\frac{\partial \rho_{\text{TOTAL}}}{\partial t}.$$

HE FELT WE SHOULD ADD YET ANOTHER TERM SO THAT $\bar{\nabla}\cdot\bar{J}_{\text{TOTAL}} = 0$. TOWARDS THIS END, RECALL THAT

$$\bar{\nabla}\cdot\bar{E} = 4\pi \rho_{\text{TOTAL}}, \quad \text{so} \quad \frac{\partial \rho_{\text{TOTAL}}}{\partial t} = \frac{1}{4\pi}\bar{\nabla}\cdot\frac{\partial \bar{E}}{\partial t}.$$

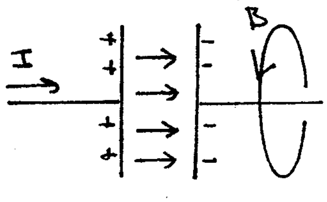
SO WE DEFINE $\bar{J}_{\text{TOTAL}} = \bar{J}_{\text{FREE}} + \frac{\partial \bar{P}}{\partial t} + c\,\bar{\nabla}\times\bar{M} + \frac{1}{4\pi}\frac{\partial E}{\partial t}$,

WHICH OBEYS $\bar{\nabla}\cdot\bar{J}_{\text{TOTAL}} = 0$. WITH $\bar{D} = \bar{E} + 4\pi\bar{P}$ WE CAN WRITE

$$\bar{J}_{\text{TOTAL}} = \bar{J}_{\text{FREE}} + \frac{1}{4\pi}\frac{\partial \bar{D}}{\partial t} + c\,\bar{\nabla}\times\bar{M}.$$

THE NEW PIECE $\frac{1}{4\pi}\frac{\partial \bar{D}}{\partial t} \equiv$ <u>DISPLACEMENT</u> <u>CURRENT</u>.

<u>EXAMPLE</u> MAGNETIC FIELD IN A CHARGING CAPACITOR



BY AMPERE'S LAW, LINES OF $\bar{B}$ CIRCULATE ABOUT THE WIRE WITH

$$\oint \bar{B}\cdot d\bar{l} = \frac{4\pi}{c}\int_{\text{SURFACE OF LOOP}} \bar{J}\cdot d\bar{S}, \quad \text{so} \quad B = \frac{2I}{rc}.$$

BUT SUPPOSE WE TAKE THE SURFACE OF THE LOOP TO PASS BETWEEN THE CAPACITOR PLATES. THEN $I_{\text{THRU}} = 0$. BUT DOES $\bar{B}$ SUDDENLY VANISH JUST BECAUSE WE CHANGE OUR SURFACE? — THE LOOP BOUNDING THE SURFACE HAS NOT CHANGED!

EVERYTHING IS FINE IF WE CONSIDER THE DISPLACEMENT CURRENT. BETWEEN THE PLATES, WE HAVE ELECTRIC FIELD $E = \frac{4\pi Q}{A}$ (A = AREA)

SO $\frac{1}{4\pi}\frac{\partial D}{\partial t} = \frac{\dot{Q}}{A} = \frac{I}{A}$ AND $\int \frac{1}{4\pi}\frac{\partial \bar{D}}{\partial t}\cdot dS = I$

NOTE THAT WE ALSO EXPECT A CIRCULATING MAGNETIC FIELD BETWEEN THE PLATES.

THE MAGNETIC FIELD $\bar{H}$

WE RETURN TO MAGNETOSTATICS, BUT EXTEND OUR DISCUSSION TO INCLUDE MATERIALS WITH MAGNETIZATION DENSITY $\bar{M}$, AS WELL AS CASES WITH FREE CURRENTS $\bar{J}_{\text{FREE}}$.

THEN $\nabla\cdot\bar{B} = 0 \qquad \bar{\nabla}\times\bar{B} = \frac{4\pi}{c}\bar{J}_{\text{FREE}} + 4\pi\,\bar{\nabla}\times\bar{M}$

SUPPOSING $\bar{J}_{\text{FREE}}$ TO BE KNOWN, IT IS APPROPRIATE TO WRITE

$$\bar{\nabla}\times(\bar{B} - 4\pi\bar{M}) = \frac{4\pi}{c}\bar{J}_{\text{FREE}}$$

WE ARE LED TO DEFINE $\underline{\bar{H} = \bar{B} - 4\pi\bar{M}}$ = MAGNETIC <u>FIELD</u>

WHERE $\bar{B}$ = MAGNETIC <u>INDUCTION</u> STRICTLY SPEAKING.

FROM OUR MICROSCOPIC VIEWPOINT, $\bar{H}$ IS AN AUXILIARY QUANTITY LIKE $\bar{D}$, WHICH IS USEFUL IN PROBLEMS INVOLVING MAGNETIC MEDIA. HISTORICALLY THE MACROSCOPIC PROBLEMS CAME FIRST, AND THE MICROSCOPIC VIEW LATER. HENCE PEOPLE OFTEN CALL $\bar{H}$ RATHER THAN $\bar{B}$ <u>THE</u> MAGNETIC FIELD.

THIS ATTITUDE REMAINS PERFECTLY JUSTIFIABLE TODAY FOR WORK DEALING WITH MAGNETIC MEDIA.

BUT THOSE WHO PREFER TO EMPHASIZE THE MICROSCOPIC VIEW WILL REGARD $\bar{B}$ AS 'MORE FUNDAMENTAL' THAN $\bar{H}$.

<u>EXAMPLE</u> DEFLECTION OF A CHARGED PARTICLE PASSING THRU MAGNETISED IRON.

SHOULD WE REPLACE THE LORENTZ FORCE LAW, $\bar{F} = e\frac{\bar{v}}{c}\times\bar{B}$

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BY $\bar{F} = q \frac{\bar{v}}{c} \times \bar{H}$ INSIDE THE IRON ? NO!

AS WE SHALL REMARK BELOW, INSIDE THE IRON OF AN ELECTROMAGNET WE MAY HAVE $\bar{B} \sim 1000 \bar{H}$. THE EXPERIMENTALLY OBSERVED DEFLECTION IS CONSISTENT WITH THE LARGE VALUE OF $\bar{B}$.

THE DIFFERENTIAL EQUATIONS FOR $\bar{H}$ ARE [MAGNETOSTATICS]

$$\bar{\nabla} \times \bar{H} = \frac{4\pi}{c} \bar{J}_{FREE} \quad \text{AND} \quad \bar{\nabla} \cdot \bar{H} = -4\pi \, \bar{\nabla} \cdot \bar{M}$$

IF MAGNETIC CHARGES (MONOPOLES) EXISTED, WE RECOGNIZE THAT $-\bar{\nabla} \cdot \bar{M} = \rho_M =$ MAGNETIC CHARGE DENSITY

THEN WE COULD WRITE $\bar{\nabla} \cdot \bar{H} = 4\pi \rho_M$. EVEN IF MONOPOLES DON'T EXIST, WE CAN STILL WRITE THIS, BUT EXPECT $\rho_M = 0$ IN BULK, AND ONLY AT SURFACES WILL A $\sigma_M \neq 0$ DEVELOP FROM DIPOLES....

THE MAGNETIC SCALAR POTENTIAL

IN PROBLEMS WHERE $\bar{J}_{FREE} = 0$, BUT NON-ZERO MAGNETISATION EXISTS, WE HAVE $\bar{\nabla} \times \bar{H} = 0 \qquad \bar{\nabla} \cdot \bar{H} = 4\pi \rho_M$

THE EQUATIONS FOR $\bar{H}$ NOW HAVE SOME RESEMBLANCE TO THOSE OF ELECTROSTATICS. IN PARTICULAR, $\bar{\nabla} \times \bar{H} = 0$, SUGGESTS WE COULD WRITE

$$\bar{H} = -\bar{\nabla} \phi_M \quad \text{WHERE } \phi_M = \underline{\text{MAGNETIC SCALAR POTENTIAL}}$$

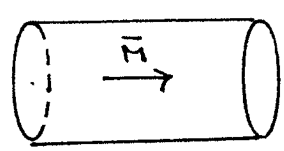
A FORMAL SOLUTION FOR ϕ_M IS (C.F. p16)

$$\phi_M = \int \frac{\rho_M}{r} \, dvol + \int_{SURFACE} \frac{\bar{M} \cdot d\bar{S}}{r}$$

THIS IS TO BE COMPARED WITH THE FORMAL SOLUTION FOR $\bar{B} = \bar{\nabla} \times \bar{A}$

$$\text{WITH } \bar{A} = \frac{1}{c} \int \frac{c \, \bar{\nabla} \times \bar{M}}{r} \, dvol + \frac{1}{c} \int_{SURFACE} \frac{c \bar{M} \times d\bar{S}}{r} \qquad \text{(SEE p. 91)}$$

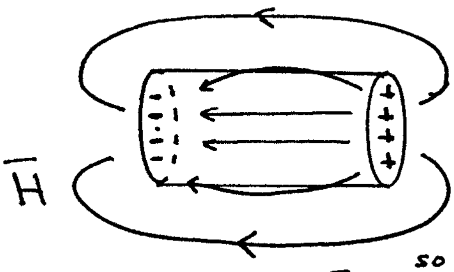
<u>EXAMPLE</u> A PERMANENTLY MAGNETISED CYLINDER, OF UNIFORM MAGNETISATION DENSITY $\bar{M}$.



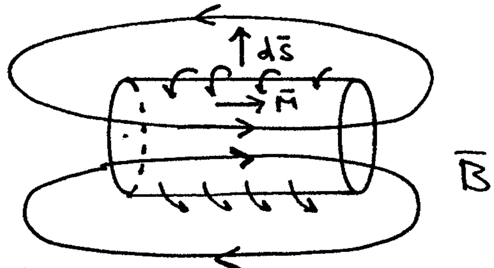
INSIDE THE CYLINDER $\rho_M = -\bar{\nabla} \cdot \bar{M} = 0$, AND $\bar{\nabla} \times \bar{M} = 0$

BUT $\bar{M} \cdot d\bar{S} \neq 0$ ON THE ENDS $\Leftrightarrow$ SURFACE POLE DENSITY, $\sigma_M = M$ PER UNIT AREA

AND $\bar{M} \times d\bar{S} \neq 0$ ON THE SIDES $\Rightarrow$ SURFACE CURRENT DENSITY, $\lambda = cM$ PER UNIT LENGTH



$\bar{B} = \bar{H}$ OUTSIDE.
IF CYLINDER LONG & THIN
$\bar{B}_{IN} \sim 4\pi \bar{M}$
$\Rightarrow \bar{H}_{IN} \sim 0$
BUT IF CYLINDER IS SHORT AND FAT
$\bar{H}_{IN} \sim -4\pi \bar{M}$
$\Rightarrow \bar{B}_{IN} \sim 0$
SO LONG & THIN GIVES MORE EXTERNAL MAG FIELD.



SOLUTION FOR $\bar{H}$ VIA THE SCALAR POTENTIAL

SOLUTION FOR $\bar{B}$ VIA VECTOR POTENTIAL (& AMPERE'S LAW)

THE RELATIVE SIMPLICITY OF THE SOLUTION VIA THE MAGNETIC SCALAR POTENTIAL LED PEOPLE TO TRY TO EXTEND IT TO CASES WHERE FREE CURRENTS ARE PRESENT, BUT THERE IS NO MAGNETISATION. THEN $\bar{B} = \bar{H}$, AND $\bar{\nabla} \times \bar{B} = 0$ OUTSIDE REGIONS WHERE CURRENTS FLOW. SO PERHAPS WE CAN WRITE

$$\bar{B} = -\nabla \phi_M \quad \text{IN REGIONS WHERE } \bar{J}_{FREE} = 0$$

HOWEVER THIS CANNOT BE DONE BY A <u>SINGLE-VALUED</u> FUNCTION ϕ_M.

SEE BECKER SEC. 46 FOR A SKETCH OF HOW ϕ_M DUE TO A LOOP OF CURRENT IS CALCULATED BY SUPPOSING THE SURFACE BOUNDED BY THE LOOP IS COVERED WITH A LAYER OF MAGNETIC DIPOLES .....

MAGNETIC MATERIALS (BECKER SEC 48)

THERE ARE SEVERAL TYPES OF MATERIALS WHICH CAN POSSESS A MAGNETISATION DENSITY - WHICH IS DUE TO SOME KIND OF 'ATOMIC OR MOLECULAR' CURRENT LOOPS WITHIN THE MATERIAL. THIS BEHAVIOR CANNOT BE WELL-ACCOUNTED FOR IN CLASSICAL DESCRIPTIONS, SO WE ONLY MENTION A FEW BASIC FACTS.

<u>DIAMAGNETIC MATERIALS</u> WHEN A FIELD $\bar{B}$ IS APPLIED TO SUCH MATERIALS, CHANGES IN THE MOTION OF THE ATOMIC ELECTRONS ARE INDUCED, SO AS TO GIVE A NET MAGNETISATION. BY LENZ'S LAW (LECTURE 9), THE FIELD OF THE INDUCED MAGNETISATION WILL OPPOSE THAT OF THE FIELD $\bar{B}$ WHICH CREATES IT. REFERING TO THE PICTURE AT THE TOP OF THE PAGE, THIS MEANS

$$\bar{M} = -\alpha \bar{B}_{EXTERNAL}$$

WHERE α IS EMPIRICALLY FOUND TO BE INDEPENDENT OF B. (FOR B 'WEAK')

THEN $\bar{H} = \bar{B} - 4\pi \bar{M} = (1 + 4\pi\alpha)\bar{B}$

IT IS CUSTOMARY TO WRITE $\bar{B} = \mu \bar{H}$ WHERE μ = <u>PERMEABILITY</u>

THEN $\mu < 1$ FOR DIAMAGNETISM.

ESSENTIALLY ALL MATERIALS EXHIBIT DIAMAGNETISM, ALTHOUGH TYPICALLY $1-\mu$ IS VERY SMALL.

<u>PARAMAGNETIC MATERIALS</u> THESE MATERIALS CONTAIN ATOMS WITH MAGNETIC MOMENTS OF FIXED MAGNITUDE.

NORMALLY THE DISTRIBUTION OF DIRECTIONS OF THE MOMENTS IS RANDOM, SO THERE IS NO NET MAGNETISATION. BUT IF A FIELD IS APPLIED, THE MOMENTS TEND TO LINE UP ALONG THE FIELD TO MINIMISE THE ENERGY. AS DISCUSSED IN LECTURE 2, THERMAL ENERGY TRANSFERS PREVENT COMPLETE ALIGNMENT, AND WE EXPECT

$$\bar{M} \approx \frac{N m^2 \bar{B}}{3KT} \qquad m = \text{ATOMIC MOMENT} \quad T = \text{TEMPERATURE}$$

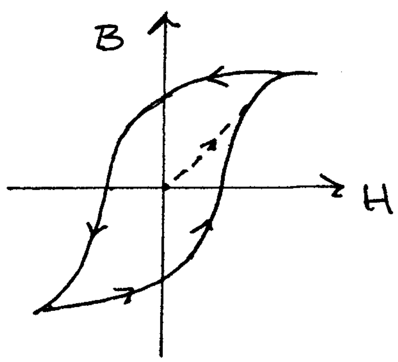
AT LARGE T.

NOW $\bar{H} = \bar{B} - 4\pi\bar{M} < \bar{B}$ SO IF $\bar{B} = \mu\bar{H}$, $\mu > 1$.

[FOR SOMEWHAT MORE DETAIL, SEE FEYNMAN RED BOOKS, VOL II., CHAPS. 34-37.]

<u>FERROMAGNETIC MATERIALS</u> IN THESE MATERIALS, MAINLY IRON, NICKEL AND COBALT, INTERACTIONS BETWEEN THE INTRINSIC MOMENTS OF ELECTRONS CAUSE THESE MOMENTS TO LINE UP EVEN IN THE ABSENCE OF AN EXTERNAL FIELD. THE ALIGNMENT MAY EXTEND OVER ONLY A LIMITED VOLUME, CALLED A 'DOMAIN', WHILE THE NEIGHBORING DOMAIN HAS A DIFFERENT DIRECTION TO ITS ALIGNMENT. IF ONE DOMAIN DOMINATES OVER ALL THE OTHERS WE HAVE A 'PERMANENT' MAGNET.

IF AN EXTERNAL MAGNETIC FIELD IS APPLIED, THE DOMAINS ALIGNED ALONG THE FIELD GROW WHILE OTHERS SHRINK. THIS PROCESS IS NOT VERY LINEAR, AND IS EMPIRICALLY DESCRIBED BY THE <u>HYSTERESIS CURVE</u> OF B VS H



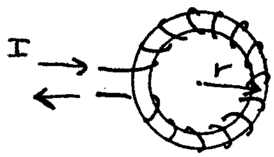
IF WE START FROM $B = H = 0$, THE INITIAL GROWTH OF B WITH H CAN BE VERY FAST. $\mu_{EFF} = \frac{B}{H}$ CAN BE > 10000.

BUT EVENTUALLY 'SATURATION' TAKES PLACE — ALL MOMENTS ARE ALIGNED, AND TYPICALLY $B \sim 20000$ GAUSS — THE CGS UNIT OF FIELD STRENGTH.

(IF WE INCREASED H BEYOND 20000 GAUSS, B WOULD RISE LINEARLY WITH H AS IF $\mu = 1$)

EXAMPLES OF MAGNETOSTATICS PROBLEMS

1. TOROIDAL ELECTROMAGNET



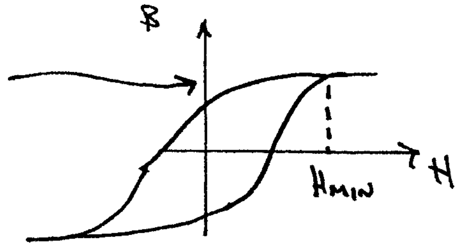
AN IRON RING (TOROID) IS WRAPPED WITH N TURNS OF WIRE. CURRENT I IS PASSED THRU THE WIRE.

CONSIDER A CLOSED LOOP INSIDE THE RING OF RADIUS r.

AMPERE'S LAW TELLS US $\oint \bar{H}\cdot d\bar{l} = \frac{4\pi}{c}\int \bar{J}\cdot d\bar{S}$

OR $2\pi r H = \frac{4\pi NI}{c} \Rightarrow H = \frac{2NI}{rc}$

THEN $B = \mu H$ CAN BE DETERMINED FROM THE HYSTERESIS CURVE.

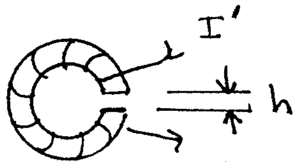


IF $H > H_{MIN}$, THEN $B = B_{SATURATION}$ EVERYWHERE INSIDE THE RING.

i.e. $B \approx$ CONSTANT ALTHOUGH $H \sim \frac{1}{r}$

THIS MAGNET IS NOT VERY USEFUL AS IS!

2. IRON MAGNET WITH AN AIR GAP



SUPPOSE WE WOULD LIKE TO OBTAIN THE SAME FIELD STRENGTH B IN THE GAP OF HEIGHT h, AS WE HAD IN THE PREVIOUS EXAMPLE INSIDE THE IRON. THIS WILL REQUIRE A DIFFERENT CURRENT I'.

WE NEED THE BOUNDARY CONDITIONS AT THE GAP.

$\bar{\nabla}\cdot\bar{B} = 0 \Rightarrow B_\perp$ IS CONTINUOUS

BUT $B = \mu H$, SO $\underbrace{\mu_{AIR}}_{=1} H_{GAP} = \mu_{IRON} H_{IRON}$

WE WANT $H_{GAP} = B_{GAP} = B_{\text{OF EXAMPLE 1}} = \frac{2\mu NI}{rc} \Rightarrow H_{IRON} = \frac{2NI}{rc}$

AMPERE $\Rightarrow \frac{4\pi NI'}{c} = \oint \bar{H}\cdot d\bar{l} = (2\pi r - h)\frac{2NI}{rc} + h\cdot\frac{2\mu NI}{rc} = \frac{4\pi NI}{c}\left(1 + \frac{(\mu-1)h}{2\pi r}\right)$

SO $I' = I\left(1 + \frac{(\mu-1)h}{2\pi r}\right)$

WE USE IRON WITH HIGH PERMEABILITY TO GET A LOT OF B FOR ONLY A LITTLE I. SUPPOSE $\mu \sim 1000$.

THEN EVEN IF $h/2\pi r \sim \frac{1}{1000}$ (A VERY SMALL GAP)

WE WOULD HAVE $I' = 2I$. SO WHAT?

IF THE WIRE WINDING HAS RESISTANCE R, IT CONSUMES POWER I^2R DUE TO JOULE HEATING. HENCE OUR TINY AIR GAP MULTIPLIES THE POWER BILL BY 4. WE SEE THAT ROUGHLY POWER $\sim h^2$

IT IS VERY EXPENSIVE TO OPERATE BIG AIR-GAP ELECTROMAGNETS!

3. MAGNETIC CIRCUIT ANALOGY

EXAMPLES 1 AND 2 CAN BE CAST INTO A FORM WHICH RESEMBLES ELECTRICAL CIRCUIT PROBLEMS INVOLVING OHM'S LAW.

$\bar{j} = \sigma \bar{E} \quad \longleftrightarrow \quad \bar{B} = \mu \bar{H}$

$\oint \bar{E} \cdot d\bar{l} = \text{EMF} \qquad \oint \bar{H} \cdot d\bar{l} = \frac{4\pi NI}{c}$

$\bar{\nabla} \cdot \bar{j} = 0 \qquad \bar{\nabla} \cdot \bar{B} = 0$

$\text{EMF} = I \cdot \text{RESISTANCE} \qquad \frac{4\pi NI}{c} = \text{FLUX} \cdot \text{RELUCTANCE}$

$I = j A \qquad \text{FLUX} = B \cdot \text{AREA} \quad (\perp \text{ TO } B)$

$R = \frac{\text{LENGTH}}{\sigma \text{ AREA}}$ = RESISTANCE $\longleftrightarrow$ $R = \frac{\text{LENGTH}}{\mu \cdot \text{AREA}}$ (= RELUCTANCE)

LET'S TRY IT.

1'. THE SOLID IRON RING: $\frac{4\pi NI}{c} = B \cdot \text{AREA} \cdot \frac{\text{LENGTH}}{\mu \cdot \text{AREA}} = \frac{2\pi r B}{\mu}$

2' RING WITH A GAP: WE ADD THE TWO RELUCTANCES IN SERIES

$$\frac{4\pi NI'}{c} = B \cdot \text{AREA} \, (R_1 + R_2)$$
$$= B \cdot \text{AREA} \left(\frac{2\pi r - h}{\mu A} + \frac{h}{A} \right)$$
$$= \frac{2\pi r B}{\mu} \left(1 + \frac{(\mu - 1) h}{2\pi r} \right)$$

4. UNIFORMLY MAGNETISED SPHERE

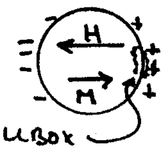
$\bar{m}$ = MAGNETISATION DENSITY

A SOLUTION FOR $\bar{H}$ EMPHASIZES THE SIMILARITY TO ELECTROSTATICS, SINCE $\bar{\nabla}\cdot\bar{H} = -4\pi\bar{\nabla}\cdot\bar{m} = 4\pi\rho_M$

HENCE WE MAY TAKE OVER THE RESULTS OF THE CASE OF A UNIFORMLY POLARIZED DIELECTRIC SPHERE.

OUTSIDE, THE FIELD $\bar{B}=\bar{H}$ IS JUST THAT DUE TO THE TOTAL DIPOLE MOMENT $\bar{M} = \frac{4\pi a^3}{3}\bar{m}$

INSIDE $\bar{H}$ (THE EQUIVALENT OF $\bar{E}$) IS CONSTANT AND OPPOSITE TO $\bar{m}$



SUPPOSE WE FORGOT THE NUMERICAL RELATION BETWEEN $\bar{E}_{INSIDE}$ AND $\bar{P}$, BUT RECALL THAT $\bar{E}_{IN}$ IS CONSTANT....

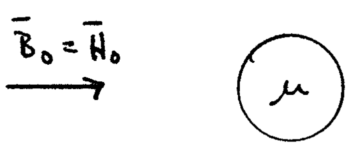
PILLBOX

WE COULD CONSIDER A SMALL GAUSSIAN PILLBOX ON THE AXIS AT THE SURFACE OF THE SPHERE, WHERE $\sigma_M = \bar{m}\cdot\hat{n} = m$

THEN BY GAUSS' LAW $4\pi\sigma_M = H_{OUT} - H_{IN}$

BUT $\bar{H}_{OUT} = \frac{3(\bar{M}\cdot\hat{r})\hat{r} - \bar{M}}{r^3}$ GENERALLY, SO $H_{OUT} = \frac{2M}{a^3} = \frac{8\pi}{3}m$ AT THE PILLBOX. HENCE $\bar{H}_{IN} = \frac{8\pi\bar{m}}{3} - 4\pi\bar{m} = -\frac{4\pi\bar{m}}{3} \Rightarrow \bar{B}_{IN} = \frac{8\pi\bar{m}}{3}$

5. PERMEABLE SPHERE IN AN EXTERNAL FIELD



AGAIN WE RECOGNIZE THE EQUIVALENCE TO THE PROBLEM OF A DIELECTRIC SPHERE IN AN EXTERNAL FIELD.

PERHAPS ALL WE RECALL OF THE SOLUTION IS THAT INSIDE THE SPHERE, $\bar{B}$, $\bar{H}$ AND THE INDUCED MAGNETISATION $\bar{m}$ WILL BE UNIFORM. THIS SHOULD BE ENOUGH, IF WE BUILD ON EXAMPLE 4.

WE EXPECT $\bar{H}_{IN} = \bar{H}_0 - \frac{4\pi}{3}\bar{m}$ & $B_{IN} = \bar{B}_0 + \frac{8\pi}{3}\bar{m}$

BUT $B_{IN} = \mu H_{IN}$ FOR A PERMEABLE SPHERE, SO

$$\left(B_0 + \frac{8\pi m}{3}\right) = \mu\left(\underbrace{H_0}_{=B_0} - \frac{4\pi m}{3}\right)$$

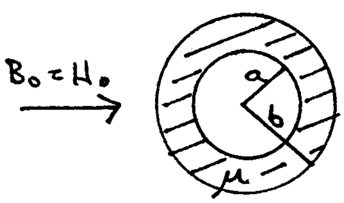
LEADING TO $\bar{m} = \frac{3}{4\pi}\frac{\mu-1}{\mu+2}\bar{B}_0$ ETC. OUTSIDE, JUST ADD THE FIELD OF DIPOLE $\frac{4\pi a^3\bar{m}}{3}$ TO $\bar{B}_0$ ...

6. MAGNETIC SHIELDING BY A PERMEABLE CYLINDER

WE NOTED THAT IT IS POSSIBLE TO SHIELD A VOLUME FROM EXTERNAL ELECTRICAL EFFECTS BY SURROUNDING THE VOLUME WITH A CONDUCTING SURFACE. CAN WE PROVIDE A SIMILAR SHIELDING FROM MAGNETIC EFFECTS? YES, IF WE SURROUND THE VOLUME WITH A SUPERCONDUCTOR - A MATERIAL WITH ABSOLUTELY NO RESISTANCE (CONDUCTIVITY $\to \infty$). THIS SOLUTION IS BECOMING MORE AND MORE PRACTICAL....

FOR NOW WE CONSIDER A CLASSICAL METHOD OF OBTAINING AT LEAST PARTIAL SHIELDING. SURROUND THE VOLUME WITH A SURFACE OF HIGHLY PERMEABLE MAGNETIC MATERIAL.

WE CONSIDER THE EFFECT OF A PERMEABLE CYLINDER ORIENTED WITH ITS AXIS $\perp$ TO THE EXTERNAL MAGNETIC FIELD



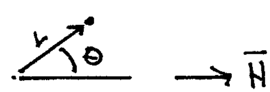
WE WANT A SOLUTION ONLY IN THE REGION AROUND THE CYLINDER, WHICH IS FAR FROM THE CURRENTS WHICH PRODUCE THE EXTERNAL FIELD.

THEN $\bar\nabla \times \bar H = 0 \Rightarrow$ WE CAN WRITE $\bar H = -\nabla \phi_M$

ALSO $\bar\nabla \cdot \bar B = 0$ AND $\bar B = \mu \bar H$, SO $\bar\nabla \cdot \bar H = 0$ IN REGIONS OF CONSTANT μ. HENCE

$$\bar\nabla \cdot \bar H = -\nabla^2 \phi_M = 0 \quad \text{IN EACH REGION}$$

THIS IS LAPLACE'S EQUATION, WHICH WE HAVE SOME EXPERIENCE AT SOLVING. WE WRITE DOWN THE GENERAL SOLUTIONS IN EACH OF THE 3 REGIONS, AND MATCH BOUNDARY CONDITIONS AT $r = a$ AND $r = b$.



AS IN LECTURE 5, WE EXPECT

$$\phi_M = (E + F\theta)(G + H \ln r) + \sum_n \left(A_n \cos k_n\theta + B_n \sin k_n \theta\right)\left(C_n r^{k_n} + \frac{D_n}{r^{k_n}}\right)$$

FOR OUR GEOMETRY WE EXPECT $\phi(-\theta) = \phi(\theta)$

AND $\phi(\theta + 2\pi) = \phi(\theta)$

$$\text{SO } \phi_M = \sum_n \cos n\theta \left(A_n r^n + \frac{B_n}{r^n}\right)$$

ALSO AS $r \to \infty$, $\quad \bar H \to \bar H_0 \Rightarrow \phi_M \to -H_0 x = -H_0 r \cos\theta$

HENCE FOR $r > b$

$$\phi_M = -H_0 r \cos\theta + \sum_n \frac{A_n \cos n\theta}{r^n}$$

$a < r < b$ $\quad \phi_m = \sum_n \cos n\theta \left(B_n r^n + \frac{C_n}{r^n} \right)$

$r < a$ $\quad \phi_m = \sum_n D_n \cos n\theta \, r^n$

WE EXPECT ϕ_m CONTINUOUS AT $r = a$ AND b.

ALSO $\bar{\nabla} \cdot \bar{B} = 0 \Rightarrow B_\perp$ CONTINUOUS $\Rightarrow \mu H_\perp$ CONTINUOUS

AND $H_\perp = -\frac{\partial \phi_m}{\partial r}$

THIS DETERMINES THE EXPANSION TO HAVE NON-VANISHING COEFFICIENTS ONLY FOR $n = 1$, AND IN PARTICULAR

$$D_1 = \frac{-4\mu H_0}{(\mu+1)^2 - \frac{a^2}{b^2}(\mu-1)^2}$$

FOR $\mu \gg 1$, $\quad D_1 = \frac{-4 H_0}{\mu (1 - a^2/b^2)}$

AND SO $\quad \frac{B_{IN}}{B_0} = \frac{4}{\mu (1 - a^2/b^2)}$

FOR EXAMPLE, WITH $\mu \sim 10{,}000$ AND $a \sim .95 b$

WE FIND $\quad B_{IN} \sim \frac{1}{250} B_0$