Maxwellian Vacuum Polarization

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1 Problem

Maxwell formulated his dynamical theory of the electromagnetic field [1] without a crisp vision of the nature of electric charge. The notion that electric charge resides on (rather than, say, in the space/æther outside) "point" particles became widely accepted only after the great 1892 monograph of Lorentz [2]. Nonetheless, Maxwell's equations are consistent with the view that "free" charges and currents do not exist, and that all charges and currents are related to "bound" electric and magnetic polarization densities $\mathbf{P}$ and $\mathbf{M}$ in the æther/vacuum according to

$$\rho_{\rm total} = -\boldsymbol{\nabla}\cdot\mathbf{P}, \qquad \mathbf{J}_{\rm total} = \frac{\partial \mathbf{P}}{\partial t} + c\boldsymbol{\nabla}\times\mathbf{M}, \tag{1}$$

in Gaussian units, where $\rho_{\rm total}$ and $\mathbf{J}_{\rm total}$ are the densities of (bound = total) electric charge and current, respectively.

Discuss the bound polarizations associated with a "point" electric charge q, and "point" electric and magnetic dipoles $\mathbf{p}$ and $\mathbf{m}$, which may be in motion.

2 Solution

In the convention of eq. (1), Maxwell's equations can be written as

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho_{\rm total} = -4\pi\boldsymbol{\nabla}\cdot\mathbf{P}, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{3}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi\mathbf{J}_{\rm total}}{c} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 4\pi\boldsymbol{\nabla}\times\mathbf{M} + \frac{1}{c}\frac{\partial(\mathbf{E}+4\pi\mathbf{P})}{\partial t}, \tag{5}$$

If we make the usual definitions of the auxiliary fields,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \qquad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}, \tag{6}$$

then

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \tag{7}$$

as expected since there are no "free" charges or currents by definition.

If $\rho_{\rm total}$ and $\mathbf{J}_{\rm total}$ are known then the equations (2)-(5) have formal solutions for $\mathbf{E}$ and $\mathbf{B}$, since both the curl and divergences of these fields are specified. However, eq. (1) is not sufficient to determine $\mathbf{P}$ and $\mathbf{M}$ uniquely, as the curl of $\mathbf{P}$ and the divergence of $\mathbf{M}$ are not known. In general, the polarization densities are not well defined, and only become so upon imposition of some auxiliary condition(s).

In this note we show that an assumption of spherical symmetry is sufficient to determine the electric-polarization density associated with a point charge at rest, and that "simple" additional assumptions permit determination of the polarization densities for electric and magnetic dipoles.

2.1 Point Charge q

We first consider a point charge q at rest, in vacuum, at the origin. The electric field in this rest frame is

$$\mathbf{E}_0 = \frac{q\,\hat{\mathbf{r}}_0}{r_0^2}\,. \tag{8}$$

This case is spherically symmetric, so we expect the electric polarization $\mathbf{P}_0$ to be spherically symmetric also, such that the integral form of eq. (2) yields

$$\mathbf{P}_0 = -\frac{q\,\hat{\mathbf{r}}_0}{4\pi r_0^2} = -\frac{\mathbf{E}_0}{4\pi}\,. \tag{9}$$

Thus, $\mathbf{D}_0 = \mathbf{E}_0 + 4\pi\mathbf{P}_0 = 0$ here (except at the origin where the fields are not defined). Of course, there is zero magnetic polarization in this case,

$$\mathbf{M}_0 = 0. \tag{10}$$

The relativistic transformations of densities $\mathbf{P}$ and $\mathbf{M}$ were first discussed by Lorentz [4], who noted that they follow the same transformations as do the magnetic and electric fields $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ and $\mathbf{E} = \mathbf{D} - 4\pi\mathbf{P}$, respectively,

$$\mathbf{P} = \gamma\left(\mathbf{P}_0 + \frac{\mathbf{v}}{c}\times\mathbf{M}_0\right) - (\gamma-1)(\hat{\mathbf{v}}\cdot\mathbf{P}_0)\hat{\mathbf{v}}, \quad \mathbf{M} = \gamma\left(\mathbf{M}_0 - \frac{\mathbf{v}}{c}\times\mathbf{P}_0\right) - (\gamma-1)(\hat{\mathbf{v}}\cdot\mathbf{M}_0)\hat{\mathbf{v}}, \tag{11}$$

where the inertial lab frame moves with velocity $\mathbf{v}$ with respect to the (inertial) rest frame of the polarization densities, and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Thus, in a lab frame where charge q moves with constant, low, velocity $\mathbf{v}$ its associated polarization densities are, at the instant the charge is at the origin,

$$\mathbf{P} \approx \mathbf{P}_0 + \frac{\mathbf{v}}{c}\times\mathbf{M}_0 = -\frac{q\,\hat{\mathbf{r}}}{4\pi r^2}\,, \qquad \mathbf{M} \approx \mathbf{M}_0 - \frac{\mathbf{v}}{c}\times\mathbf{P}_0 = \frac{\mathbf{v}}{c}\times\frac{q\,\hat{\mathbf{r}}}{4\pi r^2}\,. \tag{12}$$

The polarization densities (9) and (12) are nonzero everywhere in the "empty" space around the charge. We might then characterize them as Maxwellian vacuum polarization.

The now-standard view, following Lorentz [2], that the polarization densities are zero in the "empty" space outside the charge, corresponds to the quantum view that photons carry no electric charge. This constrasts with the strong interaction, in which the massless quanta (gluons) carry strong charge (color), making the theory nonlinear. Quantum electrodynamics includes effects of vacuum polarization associated with "virtual" pairs of particle/antiparticles with opposite electric charges, which induces tiny nonlinearities that can be ignored in classical electrodynamics.

2.2 "Point" Dipoles

We consider now the case of a "point" particle that has electric and magnetic dipole moments $\mathbf{p}_0$ and $\mathbf{m}_0$ in its rest frame.

The electric dipole can be thought of as due to a pair of equal and opposite charges with small separation, and the electric field is the sum of that of the two charges,[1]

$$\mathbf{E}_0 = \frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{r}}_0)\hat{\mathbf{r}}_0 - \mathbf{p}_0}{r_0^3} - \frac{4\pi \mathbf{p}_0\, \delta^3(\mathbf{r}_0)}{3} . \tag{13}$$

If we suppose that the total electric-polarization density associated with the two charges is the sum of their polarization densities, then away from the origin,

$$\mathbf{P}_0(r > 0) = -\frac{\mathbf{E}_0(r > 0)}{4\pi} = -\frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{r}}_0)\hat{\mathbf{r}}_0 - \mathbf{p}_0}{4\pi r_0^3} . \tag{14}$$

To determine the polarization density at the origin we require that the dipole moment $\mathbf{p}_0$ be equal to the integral of the electric polarization density $\mathbf{P}_0$,

$$\mathbf{p}_0 = \int \mathbf{P}_0\, d\mathrm{Vol}. \tag{15}$$

Noting that[2]

$$\int \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{4\pi r^3}\, d\mathrm{Vol} = -\frac{\mathbf{p}}{3} , \tag{16}$$

we find

$$\mathbf{P}_0 = -\frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{r}}_0)\hat{\mathbf{r}}_0 - \mathbf{p}_0}{4\pi r_0^3} + \frac{2\mathbf{p}_0\, \delta^3(\mathbf{r}_0)}{3} . \tag{17}$$

This is very different from the usual form $\mathbf{P}_{0,\mathrm{free}} = \mathbf{p}_0\, \delta^3(\mathbf{r}_0)$ obtained by supposing that the electric polarization density of a point dipole is just the delta-function density of its dipole moment $\mathbf{p}_0$.

We suppose here that the magnetic moment $\mathbf{m}_0$ is due to a circulating electric current. If we take the view that electrodynamics in vacuum outside of the moment is the same whether the moment is due to magnetic charges or electric currents, then the magnetic polarization density has the form of eq. (17) for $r_0 > 0$, but with a sign change corresponding to the sign change in eq. (6) for the relation of the electric and magnetic polarization densities and the electromagnetic fields. If we also require that the dipole moment $\mathbf{m}_0$ be equal to the integral of the magnetic polarization density $\mathbf{M}_0$, then the latter has the form,

$$\mathbf{M}_0 = \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}}_0)\hat{\mathbf{r}}_0 - \mathbf{m}_0}{4\pi r_0^3} + \frac{4\mathbf{m}_0\, \delta^3(\mathbf{r}_0)}{3} . \tag{18}$$

[1] See, for example, eq. (4.20) of [3].
[2] See, for example, eq. (4.18) of [3].

According to eqs. (17)-(18) the fields $\mathbf{D}_0$ and $\mathbf{H}_0$ are zero except at the origin,[3] noting that for a current-loop magnetic dipole,[4]

$$\mathbf{B}_0 = \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}}_0)\hat{\mathbf{r}}_0 - \mathbf{m}_0}{r_0^3} + \frac{8\pi \mathbf{m}_0\, \delta^3(\mathbf{r}_0)}{3}. \tag{19}$$

For low velocities of the lab frame relative the to rest frame, the lab-frame polarization densities (at the instant when the particle is at the origin) follow from eqs. (17)-(18) as

$$\begin{aligned} \mathbf{P} &\approx \mathbf{P}_0 + \frac{\mathbf{v}}{c} \times \mathbf{M}_0 \\ &\approx -\frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_0}{4\pi r^3} + \frac{2\mathbf{p}_0\, \delta^3(\mathbf{r})}{3} + \frac{\mathbf{v}}{c} \times \left(\frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_0}{4\pi r^3} + \frac{4\mathbf{m}_0\, \delta^3(\mathbf{r})}{3} \right), \qquad (20) \\ \mathbf{M} &\approx \mathbf{M}_0 - \frac{\mathbf{v}}{c} \times \mathbf{P}_0 \\ &\approx \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_0}{4\pi r^3} + \frac{4\mathbf{m}_0\, \delta^3(\mathbf{r})}{3} + \frac{\mathbf{v}}{c} \times \left(\frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_0}{4\pi r^3} - \frac{2\mathbf{p}_0\, \delta^3(\mathbf{r})}{3} \right). \qquad (21) \end{aligned}$$

The meaning of moving dipole moments is ambiguous,[5] so in the lab frame we define the electric and magnetic dipole moments as integrals of the lab-frame polarization densities,

$$\mathbf{p} \equiv \int \mathbf{P}\, d\text{Vol}, \qquad \mathbf{m} \equiv \int \mathbf{M}\, d\text{Vol}. \tag{22}$$

Then, the dipole moments in the lab frame are related to those in the rest frame by

$$\mathbf{p} \approx \mathbf{p}_0 + \frac{\mathbf{v}}{c} \times \mathbf{m}_0, \qquad \mathbf{m} \approx \mathbf{m}_0 - \frac{\mathbf{v}}{c} \times \mathbf{p}_0. \tag{23}$$

However, the fields associated with an electric and/or magnetic dipole moving at low velocity are not simply the instantaneous fields of the moments $\mathbf{p}$ *and* $\mathbf{m}$*, which leads to ambiguities in interpretations of the physical significance of the dipole moments* $\mathbf{p}$ *and* $\mathbf{m}$ *in the lab frame [6, 7].*

In sum, the classical electrodynamics of Maxwell permits vacuum polarization to be prominent. While it often stated that the Lorentz force law is the needed addition to Maxwell's equation to complete the theoretical structure of classical electrodynamics, it can be underappreciated that this law includes the insight that all electric charge resides on "particles" that also have mass, such that vacuum polarization can/should be neglected. In effect, the Lorentz force law of 1892 [2] eliminated the classical æther.

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[3]In case of an electret with uniform electric polarization in its interior, one can complete the solution by assuming that the polarization is zero outside the electret (as in prob. 4.11 of [5]) or one can permit vacuum polarization outside such that $\mathbf{D}_0 = 0$ everywhere.

[4]See, for example, eq. (5.64) of [3].

[5]See, for example, sec. 2.2 of [6].

References

[1] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865),
http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_ptrsl_155_459_65.pdf

[2] H.A. Lorentz, *La Théorie Électromagnétique de Maxwell et son Application des Corps Movants*, Arch. Nederl. Sci. exactes natur. **25** (1892),
http://physics.princeton.edu/~mcdonald/examples/EM/lorentz_theorie_electromagnetique_92.pdf

[3] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999),
http://physics.princeton.edu/~mcdonald/examples/EM/jackson_ce3_pages.pdf

[4] H.A. Lorentz, *Alte und Neue Fragen der Physik*, Phys. Z. **11**, 1234 (1910),
http://physics.princeton.edu/~mcdonald/examples/EM/lorentz_pz_11_1234_10.pdf

[5] D.J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, Upper Saddle River, New Jersey, 1999).

[6] K.T. McDonald, *Mansuripur's Paradox* (May 2, 2012),
http://physics.princeton.edu/~mcdonald/examples/mansuripur.pdf

[7] V. Hnizdo and K.T. McDonald, *Fields and Moments of a Moving Electric Dipole* (Nov. 29, 2011), http://physics.princeton.edu/~mcdonald/examples/movingdipole.pdf