“Hidden” Momentum in a Set of Circulating Rockets?

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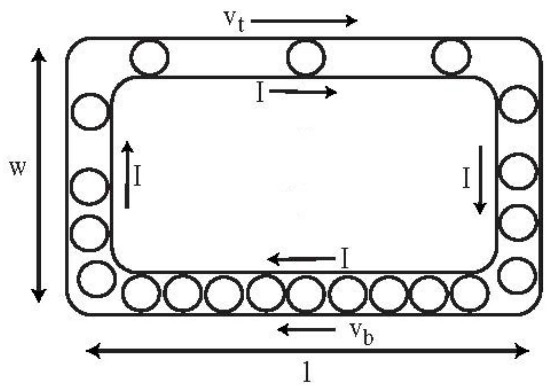
1 Problem

Discuss the momentum of a set of rockets that move in a rectangular pattern, as sketched below.



The rockets maintain constant speed on the horizontal segments in the figure, while accelerating on the left segment and decelerating on the right. The number of rockets per unit length on the various segments is such that the “rocket number current” $I = nv$ is the same on all four segments.

Does the collection of rockets contain hidden momentum, $\mathbf{P}_{\rm hidden}$, defined (when all velocities are small compared to the speed of light c) for a subsystem by

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}, \tag{1}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total “mass”, U is its total energy, $\mathbf{x}$ is its center of mass/energy, $\mathbf{v} = d\mathbf{x}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, and $\mathbf{v}_b$ is the velocity (field) of its boundary?[1]

Contrast the example of circulating rockets with the case of a circulating electrical current in an external electric field, as first considered on p. 215 of [3]. See also [4, 5, 6].

2 Solution

2.1 Circulating Rockets

The acceleration and deceleration of the rockets between the top and bottom segments of their circulating motion changes their relativistic mass.

$$m(\gamma_t - \gamma_b)c^2 = \Delta U, \tag{2}$$

[1] The definition (1) was suggested by Daniel Vanzella [1]. See also [2].

where m is the rest mass of the rockets and $\gamma = 1/\sqrt{1 - v^2/c^2}$. The "rocket current" on the top and bottom segments is

$$I = n_b v_b = n_t v_t, \tag{3}$$

such that the total momentum of the rockets is

$$\mathbf{P}_{\text{rockets}} = (n_t l m \gamma_t v_t - n_b l \gamma_b v_b)\,\hat{\mathbf{x}} = -\frac{l I \Delta U}{c^2}\,\hat{\mathbf{x}}, \tag{4}$$

taking the top and bottom segments to be in the x-direction and to have length l.

The center-of-mass velocity of the system of rockets is given by

$$M_{\text{rockets}} \mathbf{v}_{\text{cm,rockets}} = (n_t l m \gamma_t v_t - n_b l \gamma_b v_b)\,\hat{\mathbf{x}} = P_{\text{rockets}}, \tag{5}$$

where M_{rockets} is the sum of the masses of all rockets.[2]

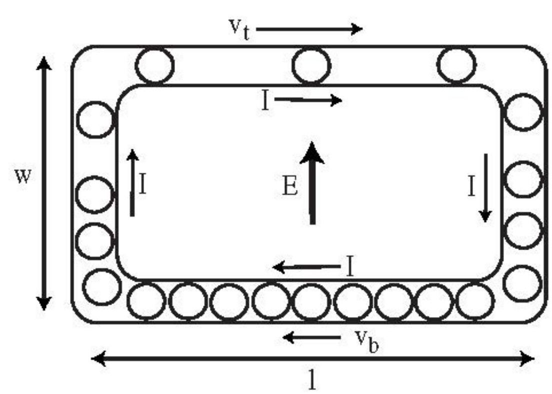
The system of rockets has no boundary, so according to the definition (1),

$$\mathbf{P}_{\text{hidden,rockets}} = \mathbf{P}_{\text{rockets}} - M_{\text{rockets}} \mathbf{v}_{\text{cm,rockets}} = 0. \tag{6}$$

That is, the system of rockets has no "hidden" momentum, although it does have small "relativistic" momentum (4) of order $1/c^2$.[3,4]

2.2 Circulating Electrical Current in an External Electric Field

This contrasts with the case where the rockets are electrical charges e constrained to circulate inside a nonconducting tube. In this case their acceleration and deceleration is due to an external electric field $\mathbf{E}$ which points upwards, as in the figure below.



[2]Although the center-of-mass velocity $\mathbf{v}_{\text{cm,rockets}}$ of the set of rockets is nonzero, the position of the center of mass of the rockets does not obey $\mathbf{x}_{\text{cm,rockets}} = \mathbf{v}_{\text{cm,rockets}} t$, in that when one rocket enters, say the "top" segment of the path, and another rocket simultaneously leaves it, the center of mass of the rockets along this segment takes a jump equal to the rocket spacing. Hence the graph of $\mathbf{x}_{\text{cm,rockets}}$ *vs.* t is a saw-tooth of slope $v_{\text{cm,rockets}}$.

[3]Although the system of circulating rockets appears to be stationary in the lab frame, it actually has nonzero momentum, and a nonzero center of mass velocity. While we might have considered these to be manifestations of "hidden" momentum, this is not the case according to the definition (1).

[4]The set of rockets is not an isolated system, in that the rockets speed up and slow down during each circuit along their common path. If the rocket exhaust particles were taken into account, then the total system would be isolated, and the overall center of mass would move with constant velocity. In the rest frame of this overall center of mass, the total momentum would be zero, in agreement with the "center-of-energy theorem" of sec. 2 of [7].

The change in a charge's energy when moving distance w through this field is

$$\Delta U = eEw = e\Delta V, \tag{7}$$

where w is the height of the loop and ΔV is the difference in the external electric potential between the bottom and the top of the loop.

We can now consider the system to contain two subsystems, the charges, and the electromagnetic fields (which include both the external electric field and the fields of the charges). It is usual to consider there to be two loops of circulating charges, which has opposite signs of charge and velocity, such that the system of charges is electrically neutral.

The electrical current of the circulating charges is given by

$$I = en_b v_b = en_t v_t. \tag{8}$$

The total momentum of the charges can now be written

$$\mathbf{P}_{\text{charge}} = (n_t l m \gamma_t v_t - n_b l \gamma_b v_b)\,\hat{\mathbf{x}} = -\frac{lI\Delta U}{c^2 e}\,\hat{\mathbf{x}} = -\frac{IlwE}{c^2 e}\,\hat{\mathbf{x}} = \frac{\mathbf{m}\times\mathbf{E}}{c}\,, \tag{9}$$

noting that the magnetic moment $\mathbf{m}$ of the loop of circulating charge is given (in Gaussian units) by

$$\mathbf{m} = \frac{I\mathbf{A}}{c} = \frac{Ilw}{c}\,\hat{\mathbf{z}}, \tag{10}$$

where $\hat{\mathbf{z}}$ is out of the paper.

The center-of-mass velocity of the charges is given by

$$M_{\text{charges}}\mathbf{v}_{\text{cm,charges}} = (n_t l m \gamma_t v_t - n_b l \gamma_b v_b)\,\hat{\mathbf{x}} = P_{\text{charges}}, \tag{11}$$

where M_{charges} is the sum of the masses of all charges. The subsystem of charges, like the system of rockets, has no "hidden" momentum according to the definition (1),

$$\mathbf{P}_{\text{hidden,charges}} = \mathbf{P}_{\text{charges}} - M_{\text{charges}}\mathbf{v}_{\text{cm,charges}} = 0. \tag{12}$$

The subsystem of the electromagnetic fields has nonzero momentum,

$$\mathbf{P}_{\text{EM}} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d\text{Vol} = \int \frac{\mathbf{J}V}{c^2}\,d\text{Vol} = \frac{\mathbf{E}\times\mathbf{m}}{c} = -\mathbf{P}_{\text{charges}}, \tag{13}$$

where the third and fourth forms of eq. (13) are due to Furry [8]. Thus, the total momentum of the system is zero, whereas the total momentum of the system of circulating rockets was nonzero.

The center of energy of the electromagnetic fields is at rest in the lab frame, so $\mathbf{v}_{\text{cm,EM}} = 0$. Hence, according to definition (1, the electromagnetic subsystem has nonzero "hidden" momentum,

$$\mathbf{P}_{\text{hidden,EM}} = \mathbf{P}_{\text{EM}} - M_{\text{EM}}\mathbf{v}_{\text{cm,EM}} = \mathbf{P}_{\text{EM}} = \frac{\mathbf{E}\times\mathbf{m}}{c} = -\mathbf{P}_{\text{charges}}, \tag{14}$$

where $M_{\text{EM}} \equiv U_{|rmEM}/c^2$.

The use of definition (1) results in a significant change in the application of the term "hidden" momentum compared to that in [4, 5]. Now, the system of circulating rockets has no "hidden" momentum, whereas it was considered that they do in [5] (sec. VI). Also, the "hidden" momentum in the system of circulating current is now associated with the electromagnetic field and not with the charges, whereas in [4, 5] the charges were considered to have "hidden" momentum while the fields did not.

References

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