

# Low-Frequency Electromagnetic Waves on a Twisted-Pair Transmission Line

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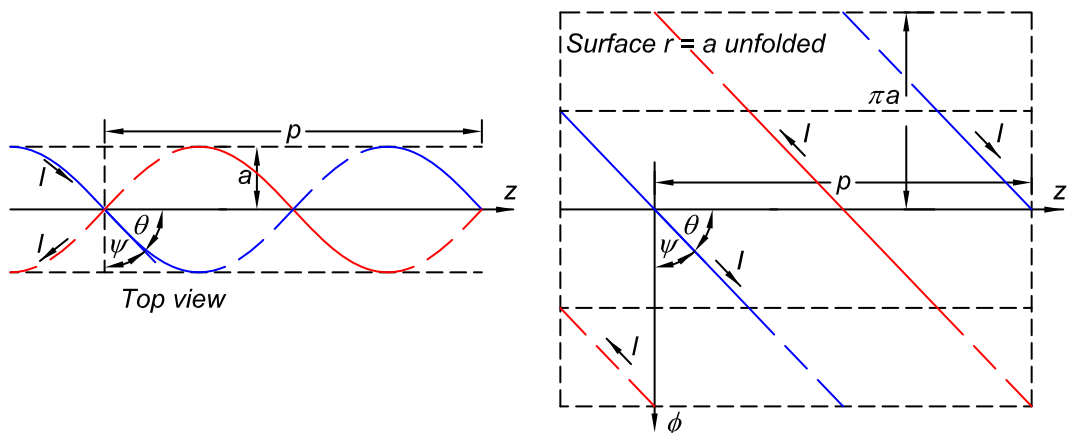
## 1 Problem

Discuss the electromagnetic waves that can propagate in the space around a transmission line whose form is a double helix of radius  $a$  and longitudinal period  $p \approx a$ . The pitch angle  $\psi$  of the helical windings with respect to the transverse planes is given by

$$\cot \psi = k_p a = \frac{2\pi a}{p}. \quad (1)$$

The angle  $\theta$  of the windings with respect to the axis of the line is then  $\theta = \pi/2 - \psi$ , *i.e.*,

$$\tan \theta = k_p a. \quad (2)$$



Such lines are extensively used for telephone communication at low frequencies for which  $ka, kp \ll 1$ , where  $k = 2\pi/\lambda = \omega/v$  is the wave number at angular frequency  $\omega$ , and  $v$  is the wave velocity. For the case that  $ka, kp \gg 1$  the waves can be thought of following the helical conductors such that the group velocity along the axis of the helix is

$$v_{g,z} \approx c \cos \theta. \quad (3)$$

Show that even at low frequencies eq. (3) is a reasonable approximation when  $a \approx p$ , but when  $a \ll p$  (a gentle twist) then  $v_{g,z} \approx c\sqrt{\cos \theta}$ .

## 2 Solution

Despite the common use of twisted-pair transmission lines, this problem seems little discussed in the literature. In the case of two-dimensional conductors there exist **transverse electromagnetic** (TEM) waves of the form  $e^{i(kz-\omega t)}$  times the (transverse) static electric and magnetic field patterns. However, TEM waves will not propagate along a twisted pair of wires, whose structure is three-dimensional.

Waves on a single helical conductor have been discussed in the context of traveling-wave amplifiers in the “sheath” approximation [1, 2], where only the part of the waves that are independent of azimuth are analyzed. A fairly general discussions of waves on twisted-pair conductors for  $ka \approx kp \approx 1$  has been given in [3], again in the context of traveling-wave amplifiers.<sup>1,2</sup>

Here, we emphasize the low-frequency behavior, when  $ka, kp \ll 1$ .

### 2.1 General Form of the Fields in Cylindrical Coordinates

We use a cylindrical coordinate system  $(r, \phi, z)$  whose axis is that of the transmission line. We ignore the insulation typically found on the wires of a twisted-pair line, and assume that the space outside the wires is vacuum.

The electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  with time dependence  $e^{-i\omega t}$  satisfy the vector Helmholtz equation,

$$(\nabla^2 + k_f^2)\mathbf{E}, \mathbf{B} = 0, \quad (4)$$

outside the wires, where

$$k_f = \frac{\omega}{c} = \frac{2\pi}{\lambda_f}. \quad (5)$$

However, in cylindrical coordinates only their  $z$ -components satisfy the scalar Helmholtz equation,

$$(\nabla^2 + k_f^2)E_z, B_z = 0. \quad (6)$$

We look for wavefunctions for  $E_z$  and  $B_z$  that propagate in the  $z$ -direction with the form

$$f_m(r)e^{-im\phi}e^{i(k_m z - \omega t)}, \quad (7)$$

where  $m$  is an integer. The (right-handed) helical conductor rotates by  $\phi = k_p z = 2\pi z/p$  as  $z$  increases, so we expect the wavefunction (7) to include this symmetry via a phase factor  $e^{-im(\phi - k_p z)}$  such that the waveform rotates as it advances. The  $z$ -dependent part of this phase contributes to the wave number  $k_m$ , which takes the form<sup>3</sup>

$$k_m = k_0(\omega) + mk_p. \quad (8)$$

<sup>1</sup>See [4] for the case of cross-wound helices.

<sup>2</sup>The magnetic fields of twisted pairs have been discussed in [5, 6, 7, 8]. Twisted-pair structures with large currents are used as undulators to generate energetic photon beams at particle accelerators (see, for example, [9]).

<sup>3</sup>The present case contrasts with that of so-called Bessel beams of order  $m$  (see, for example, the Appendix of [10]) where the drive currents are limited to a small region in  $z$ , rather than being periodic in  $z$ , such that  $k_m = k_0$  for any index  $m$ .

We are mainly interested in waves that propagate in the  $+z$  direction, for which the index  $m$  must be non-negative at low frequencies where  $0 < k_0 \ll k_p$ .<sup>4</sup>

The phase  $\varphi_m$  of the wave function (7) is  $\varphi_m = \mathbf{k}^{(m)} \cdot \mathbf{x} - \omega t = k_m z - m\phi - \omega t$ , where the wave vector  $\mathbf{k}^{(m)}$  is given by

$$\mathbf{k}^{(m)} = \nabla \varphi_m = k_m \hat{\mathbf{z}} - \frac{m}{r} \hat{\boldsymbol{\phi}}. \quad (9)$$

The phase velocity  $v_{p,m}$  of a partial wave of index  $m$  is

$$\mathbf{v}_{p,m} = \frac{\omega}{k^{(m)}} \hat{\mathbf{k}}^{(m)} = \frac{ck_f}{k_m^2 + m^2/r^2} \left( k_m \hat{\mathbf{z}} - \frac{m}{r} \hat{\boldsymbol{\phi}} \right). \quad (10)$$

We expect that  $k_0 \lesssim k_f$  ( $\ll k_p$ ) so that  $\mathbf{v}_{p,0} \lesssim c \hat{\mathbf{z}}$ , but for nonzero index  $m$  we have that  $k_m \approx mk_p$ , and hence

$$\mathbf{v}_{p,m} \approx \frac{ck_f r}{m[1 + (k_p r)^2]} \left( k_p r \hat{\mathbf{z}} - \hat{\boldsymbol{\phi}} \right), \quad (11)$$

which is small compared to  $c$  at any value of  $r$ . The wave vector  $\mathbf{k}^{(m)}$  (and the phase velocity  $\mathbf{v}_{p,m}$ ) make angle  $\theta_k$  to the  $z$ -axis given by

$$\tan \theta_k = -\frac{1}{k_p r} \quad (12)$$

for any nonzero index  $m$ . Note that at  $r = a$  the wave vector is at right angles to the direction of the helical windings, for which  $\tan \theta = k_p a$ .

The group velocity of a partial wave is<sup>5</sup>

$$\mathbf{v}_{g,m} = \nabla_{\mathbf{k}^{(m)}} \omega = \frac{\partial \omega}{\partial \mathbf{k}^{(m)}}, \quad (13)$$

whose only nonzero component is

$$v_{g,m,z} = \frac{d\omega}{dk_z^{(m)}} = \frac{d\omega}{dk_m} \approx \frac{1}{dk_m/d\omega} = \frac{1}{dk_0/d\omega} = v_{g,0,z} \equiv v_{g,z}, \quad (14)$$

independent of index  $m$ . We expect that  $v_{g,z} \lesssim c$  in the low-frequency limit.

Using eqs. (7)-(8) in the Helmholtz equation (6), we see that the radial function  $f_m$  obeys the Bessel equation

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{df_m}{dr} \right) - \left( k_m^2 - k_f^2 + \frac{m^2}{r^2} \right) f = 0, \quad (15)$$

where  $|k_m| \geq k_0 > k_f$ . The solutions to eq. (15) should remain finite at  $r = 0$  and  $\infty$ , so for  $r < a$  we use the modified Bessel function  $I_m(k'_m r)$ , and for  $r > a$  we use  $K_m(k'_m r)$ , where

$$k'_m = \sqrt{k_m^2 - k_f^2}. \quad (16)$$

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<sup>4</sup>Waves with index  $m$  negative (both for single helix and double-helix configurations) have their phase and group velocities in opposite directions. An application of such waves is the **backward wave oscillator**. See, for example, [11].

<sup>5</sup>See, for example, sec. 2.1 of [12].

That is, the longitudinal components of the electric and magnetic fields outside the wires have the forms

$$E_z(r < a) = \sum_m E_m \frac{I_m(k'_m r)}{I_m(k'_m a)} e^{-im\phi} e^{i(k_m z - \omega t)}, \quad E_z(r > a) = \sum_m E_m \frac{K_m(k'_m r)}{K_m(k'_m a)} e^{-im\phi} e^{i(k_m z - \omega t)}, \quad (17)$$

$$B_z(r < a) = \sum_m B_m \frac{I'_m(k'_m r)}{I'_m(k'_m a)} e^{-im\phi} e^{i(k_m z - \omega t)}, \quad B_z(r > a) = \sum_m B_m \frac{K'_m(k'_m r)}{K'_m(k'_m a)} e^{-im\phi} e^{i(k_m z - \omega t)}, \quad (18)$$

where  $B_m$  and  $E_m$  are constants to be determined, and  $I'_m(k'_m a) = dI_m(k'_m a)/dr$ . In eq. (17) we have noted that the Maxwell equation  $\nabla \times \mathbf{E} = ik_f \mathbf{B}$  (in Gaussian units) implies that  $E_z$  (and  $E_\phi$ ) is continuous across the surface  $r = a$ . We verify later that the normalization of coefficients  $B_m$  to  $I'_m(k'_m a)$  and  $K'_m(k'_m a)$  insures continuity of the magnetic field component  $B_r$  across this surface, as required by the Maxwell equation  $\nabla \cdot \mathbf{B} = 0$ .

The waves are driven by the current density  $\mathbf{J}$  in the twisted pair, which we can write as

$$\mathbf{J}(\mathbf{x}, t) = J(\phi, z, t) \delta(r - a) (\sin \theta \hat{\phi} + \cos \theta \hat{z}), \quad (19)$$

which points along the local direction of the twisted-pair conductors, and is confined to a thin cylinder of radius  $a$ . The wavefunction  $J(\phi, z, t)$  must have the same dependence on  $\phi$ ,  $z$  and  $t$  as eqs. (17)-(18), namely

$$J(\phi, z, t) = \sum_m J_m e^{-im\phi} e^{i(k_m z - \omega t)}, \quad (20)$$

assuming that the current only flows in the direction of the helical windings.

For a twisted pair, the current at fixed  $z$  and azimuth  $\phi + \pi$  is opposite to that at azimuth  $\phi$ , which implies that  $J_m$  is nonzero only for odd  $m$

In the case of a pair of wires of small diameter, the expansion (20) has contributions from all odd integers  $m$ . We will make a simplifying assumption that only the term  $m = 1$  is important, which corresponds to replacing the helical wires by a pair of helical wire bundles, each of which extends over  $\Delta\phi = \pi$ , such that the current in the bundles at fixed  $z$  varies as  $\cos\phi$ . If the peak current in each wire is  $I$ , then

$$J(\phi, z, t) = \frac{I}{2a \cos \theta} e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad (21)$$

$$E_z(r < a) = E_1 \frac{I_1(k'_1 r)}{I_1(k'_1 a)} e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad E_z(r > a) = E_1 \frac{K_1(k'_1 r)}{K_1(k'_1 a)} e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad (22)$$

and

$$B_z(r < a) = B_1 \frac{I'_1(k'_1 r)}{I'_1(k'_1 a)} e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad B_z(r > a) = B_1 \frac{K'_1(k'_1 r)}{K'_1(k'_1 a)} e^{-i\phi} e^{i(k_1 z - \omega t)}. \quad (23)$$

To deduce the other field components from the forms (17)-(18) it is useful to note that the electromagnetic fields can also be derived from electric and magnetic **Hertz vectors**  $\mathbf{Z}_E$  and  $\mathbf{Z}_M$  (also called **polarization potentials**; see, for example, sec. 1.11 and chap. 6 of [13]), each of which has only a  $z$ -component. These **Hertz scalars**, which we call  $Z_E$  and  $Z_M$ , obey

the scalar Helmholtz equation,  $(\nabla^2 + k_f^2)Z_E, Z_M = 0$ , outside the wires. Thus, the Hertz scalars also have the forms (22)-(23), and we will verify that

$$Z_E = -\frac{E_z}{k_1'^2}, \quad Z_M = -\frac{B_z}{k_1'^2}. \quad (24)$$

The scalar and vector potentials  $V$  and  $\mathbf{A}$  are related to the Hertz vectors according to

$$V = -\nabla \cdot \mathbf{Z}_E, \quad \mathbf{A} = \frac{1}{c} \frac{\partial \mathbf{Z}_E}{\partial t} + \nabla \times \mathbf{Z}_M, \quad (25)$$

and hence the electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{H}$  are given by

$$\mathbf{E} = \nabla(\nabla \cdot \mathbf{Z}_E) - \frac{1}{c^2} \frac{\partial^2 \mathbf{Z}_E}{\partial t^2} - \frac{1}{c} \nabla \times \frac{\partial \mathbf{Z}_M}{\partial t}, \quad \mathbf{B} = \frac{1}{c} \nabla \times \frac{\partial \mathbf{Z}_E}{\partial t} + \nabla \times (\nabla \times \mathbf{Z}_M). \quad (26)$$

The components of the electromagnetic fields in cylindrical coordinates in terms of the Hertz scalars  $Z_E$  and  $Z_M$  are (see sec. 6.1 of [13] with  $u^1 = r$ ,  $u^2 = \phi$ ,  $h_1 = 1$  and  $h_2 = r$ ),

$$E_r = \frac{\partial^2 Z_E}{\partial r \partial z} - \frac{1}{cr} \frac{\partial^2 Z_M}{\partial \phi \partial t}, \quad (27)$$

$$E_\phi = \frac{1}{r} \frac{\partial^2 Z_E}{\partial \phi \partial z} + \frac{1}{c} \frac{\partial^2 Z_M}{\partial r \partial t}, \quad (28)$$

$$E_z = -\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial Z_E}{\partial r} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial Z_E}{\partial \phi} \right) \right], \quad (29)$$

$$B_r = \frac{\partial^2 Z_M}{\partial r \partial z} + \frac{1}{cr} \frac{\partial^2 Z_E}{\partial \phi \partial t}, \quad (30)$$

$$B_\phi = \frac{1}{r} \frac{\partial^2 Z_M}{\partial \phi \partial z} - \frac{1}{c} \frac{\partial^2 Z_E}{\partial r \partial t}, \quad (31)$$

$$B_z = -\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial Z_M}{\partial r} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial Z_M}{\partial \phi} \right) \right]. \quad (32)$$

For what it's worth, the fields associated with  $Z_E$  are transverse magnetic (TM), while those associated with  $Z_M$  are transverse electric (TE).

To use the forms (22)-(23) in eqs. (27)-(32), we note that

$$I'_m(k'_m r) = k'_m I_{m-1} - \frac{m I_m}{r} = k'_m I_{m+1} + \frac{m I_m}{r}, \quad \frac{1}{r} \frac{d[r I'_m(k'_m r)]}{dr} = \left( k_m'^2 + \frac{m^2}{r} \right) I_m, \quad (33)$$

$$K'_m(k'_m r) = -k'_m K_{m-1} - \frac{m K_m}{r} = -k'_m K_{m+1} + \frac{m K_m}{r}, \quad \frac{1}{r} \frac{d[r K'_m(k'_m r)]}{dr} = \left( k_m'^2 + \frac{m^2}{r} \right) K_m, \quad (34)$$

so that for  $r < a$  the field components are

$$E_r = -\frac{1}{k_1'^2} \left[ ik_1 E_1 \frac{I'_1(k'_1 r)}{I_1(k'_1 a)} + \frac{k_f}{r} B_1 \frac{I_1(k'_1 r)}{I'_1(k'_1 a)} \right] e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad (35)$$

$$E_\phi = -\frac{1}{k_1'^2} \left[ \frac{k_1}{r} E_1 \frac{I_1(k_1' r)}{I_1(k_1' a)} - i k_f B_1 \frac{I_1'(k_1' r)}{I_1'(k_1' a)} \right] e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad (36)$$

$$E_z = -k_1'^2 Z_E = E_1 \frac{I_1(k_1' r)}{I_1(k_1' a)} e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad (37)$$

$$B_r = \frac{1}{k_1'^2} \left[ \frac{k_f}{r} E_1 \frac{I_1(k_1' r)}{I_1(k_1' a)} - i k_1 B_1 \frac{I_1'(k_1' r)}{I_1'(k_1' a)} \right] e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad (38)$$

$$B_\phi = -\frac{1}{k_1'^2} \left[ i k_f E_1 \frac{I_1'(k_1' r)}{I_1(k_1' a)} + \frac{k_1}{r} B_1 \frac{I_1(k_1' r)}{I_1'(k_1' a)} \right] e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad (39)$$

$$B_z = -k_1'^2 Z_M = B_1 \frac{I_1(k_1' r)}{I_1'(k_1' a)} e^{-i\phi} e^{i(k_1 z - \omega t)}, \quad (40)$$

and for  $r > a$  we have the forms (35)-(40) with the substitution  $I_1 \rightarrow K_1$ .

We now see that the continuity of  $E_\phi$  and  $B_r$  across the surface  $r = a$ , as previously mentioned, is satisfied by the above forms.

## 2.2 Determination of $k_0$ and the Group and Signal Velocities

The current in the helical windings is assumed to flow only at angle  $\theta$  with respect to the  $z$ -axis, so that for good conductors the conductivity of the “wires” is “infinite” in this direction, and zero in the perpendicular directions. Hence, the electric field on the surface of the cylinder  $r = a$  must be perpendicular to the direction of the current, *i.e.*,

$$E_\phi(r = a) = -\cot \theta E_z(r = a), \quad (41)$$

and hence,

$$\left( k_1'^2 a \cot \theta - k_1 \right) E_1 + i k_f a B_1 = 0. \quad (42)$$

Also, the tangential component of the magnetic field in the direction of the current must be continuous at  $r = a$ , which implies that

$$B_z(r = a_-) + \tan \theta B_\phi(r = a_-) = B_z(r = a_+) + \tan \theta B_\phi(r = a_+), \quad (43)$$

and hence,

$$i k_f a I_1'(k_1' a) K_1'(k_1' a) E_1 + \left( k_1'^2 a \cot \theta - k_1 \right) I_1(k_1' a) K_1(k_1' a) B_1 = 0. \quad (44)$$

For the simultaneous linear equations (42) and (44) to be consistent, the determinant of the coefficient matrix must vanish, *i.e.*,

$$\left( k_1'^2 a \cot \theta - k_1 \right)^2 = -(k_f a)^2 \frac{I_1'(k_1' a) K_1'(k_1' a)}{I_1(k_1' a) K_1(k_1' a)}. \quad (45)$$

This determines  $k_0$  (and therefore  $k_1$  and  $k_1'$ ) in terms of  $a$ ,  $p$  and  $k_f$ .

We restrict our attention to low frequencies such that  $k_f a \ll 1$ . *In the limit that  $k_f$  and  $k_0$  vanish, then  $k_1 = k_1' = k_p$  and  $k_p^2 a \cot \theta - k_p = 0$ , recalling that  $\cot \theta = 1/k_p a$ , so that eq. (45) is satisfied.* For small  $k_f$  and  $k_0$  we approximate

$$k_1 = k_p + k_0 \approx k_p \left( 1 + \frac{k_0}{k_p} \right), \quad k_1'^2 = k_1^2 - k_f^2 \approx k_p^2 \left( 1 + 2 \frac{k_0}{k_p} - \frac{k_f^2}{k_p^2} \right), \quad (46)$$

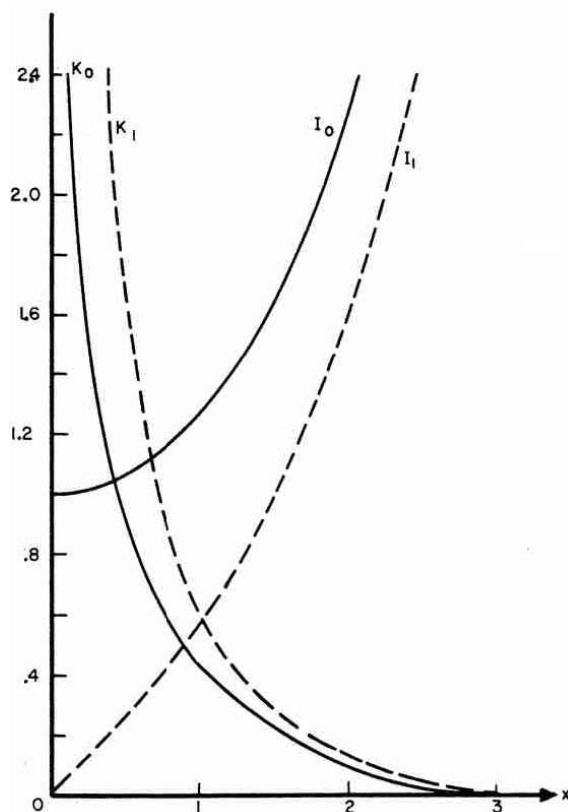
so that it suffices to take the arguments of the Bessel functions as  $k_p a$ . Using these in eq. (45) and recalling eqs. (33)-(34), we find

$$\left(k_0 - \frac{k_f^2}{k_p}\right)^2 \approx k_0^2 \approx -(k_f a)^2 \frac{I_1'(k_p a) K_1'(k_p a)}{I_1(k_p a) K_1(k_p a)} = k_f^2 C^2(k_p a), \quad (47)$$

where the constant  $C$  defined by

$$C^2(k_p a) = -a^2 \frac{I_1'(k_p a) K_1'(k_p a)}{I_1(k_p a) K_1(k_p a)} = \frac{[k_p a I_0(k_p a) - I_1(k_p a)][k_p a K_0(k_p a) + K_1(k_p a)]}{I_1(k_p a) K_1(k_p a)} \quad (48)$$

is real and positive since  $K_1'$  is negative, as seen in the figure below.



For example, if  $\theta = 45^\circ$  then  $k_p a = 1$ , and

$$C^2(1) \approx \frac{[1.2 - 0.55][0.4 + 0.6]}{0.55 \cdot 0.6} \approx 2, \quad (49)$$

and  $C(1) \approx 1.4$ .

For  $k_p a \ll 1$  (gentle twist) then  $I_0(k_p a) \approx 1 + (k_p a)^2/2$ ,  $I_1(k_p a) \approx k_p a/2 + (k_p a)^3/8$ , and  $K_1(k_p a) \gg k_p a K_0(k_p a)$ , so we have

$$C^2(k_p a \ll 1) \approx \frac{k_p a I_0(k_p a)}{I_1(k_p a)} - 1 \approx 1 + (k_p a)^2/2 \approx \frac{1}{\cos \theta}. \quad (50)$$

From eq. (47), the wave number  $k_0$  is

$$k_0 \approx C k_f = C \frac{\omega}{c}. \quad (51)$$

Recalling from eqs. (8)-(9) that  $\mathbf{k}^{(1)} \equiv \mathbf{k} = (k_0 + k_p) \hat{\mathbf{z}} - \hat{\boldsymbol{\phi}}/r$ , eq. (51) can be recast as the dispersion relation,

$$\omega = \omega(\mathbf{k}^{(1)}) \equiv \omega(\mathbf{k}) \approx \frac{c}{C} k_0 = \frac{c}{C} \left( k_z - \frac{k_p r}{r} \right) = \frac{c}{C} (k_z + k_p r k_\phi). \quad (52)$$

Then, the group velocity vector (13) is<sup>6</sup>

$$\mathbf{v}_g = \nabla_{\mathbf{k}} \omega(\mathbf{k}) = \frac{\partial \omega}{\partial k_z} \hat{\mathbf{z}} + \frac{\partial \omega}{\partial k_\phi} \hat{\boldsymbol{\phi}} \approx \frac{c}{C} (\hat{\mathbf{z}} + k_p r \hat{\boldsymbol{\phi}}). \quad (53)$$

While the  $z$ -component,  $v_{g,z}$  of the group velocity is independent of radius  $r$ , the group velocity vector  $\mathbf{v}_g$  makes angle  $\theta_g$  to the  $z$ -axis given by

$$\tan \theta_g \approx k_p r. \quad (54)$$

At very small  $r$  the group velocity is essentially parallel to the  $z$ -axis, but at large  $r$  lines of the group velocity form helices with very small pitch. The magnitude of the group velocity is

$$v_g \approx \frac{c}{C} \sqrt{1 + (k_p r)^2}, \quad (55)$$

which exceeds  $c$  at large  $r$ . However, the signal velocity  $v_s$  is clearly

$$v_s = v_{g,z} = \frac{c}{C} < c. \quad (56)$$

Comparing with eq. (12), we see that the group velocity  $\mathbf{v}_g$  is perpendicular to the phase velocity  $\mathbf{v}_p$ , and that on the surface  $r = a$  the group velocity is along the direction of the helical windings.

For  $\theta = 45^\circ$  we find that  $v_{g,z} \approx c/C \approx 0.7c \approx c \cos \theta$  for an uninsulated twisted-pair transmission line. This happens to be close to the group velocity of typical insulated, untwisted two-wire transmission lines!

For gently twisted, uninsulated pairs and low frequencies, eqs. (50) and (53) indicate that  $v_{g,z} \approx c \sqrt{\cos \theta}$ .

### 2.3 Characteristic Impedance $Z_0$ at Low Frequencies

To evaluate the characteristic impedance of the transmission line at low frequencies, we consider the radial electric field (35) for  $r < a$ , for which we need to know the constants  $B_1$  and  $E_1$  in terms of the (peak) current  $I$  in the windings.

We can relate  $B_1$  to the (peak) current  $I$  in the twisted pair via Ampère's law for a small loop of length  $dz$  in the  $r$ - $z$  plane that surrounds a short segment of the conductor where the current is maximal:

$$\begin{aligned} \frac{4\pi}{c} I_{\text{max, through loop}} &= \frac{4\pi}{c} \frac{\pi}{p} I = |B_z(r = a_-) - B_z(r = a+)| dz \\ &\approx B_1 \left( \frac{I_1(k_p a)}{I_1'(k_p a)} - \frac{K_1(k_p a)}{K_1'(k_p a)} \right) dz. \end{aligned} \quad (57)$$

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<sup>6</sup>The group velocity vector follows straight lines in homogenous media (see, for example, sec. 2.1 of [12]). Because of the twisted conductors, the present problem is not one of a homogenous medium, and the group velocity vector field need not have straight streamlines.



That is,

$$B_1 = \frac{4\pi \pi}{c p} \frac{-I'_1(k_p a) K'_1(k_p a)}{I'_1(k_p a) K_1(k_p a) - I_1(k_p a) K'_1(k_p a)} I = \frac{4\pi k_p}{c} C^2 DI, \quad (58)$$

where

$$\begin{aligned} D(k_p a) &= \frac{1}{a} \frac{I_1(k_p a) K_1(k_p a)}{I'_1(k_p a) K_1(k_p a) - I_1(k_p a) K'_1(k_p a)} \\ &= \frac{I_1(k_p a) K_1(k_p a)}{[k_p a I_0(k_p a) - I_1(k_p a)] K_1(k_p a) + I_1(k_p a) [k_p a K_0(k_p a) + K_1(k_p a)]}. \end{aligned} \quad (59)$$

Then, eqs. (42) and (51) tell us that

$$E_1 \approx -\frac{ik_f a}{k_0} B_1 \approx -\frac{ia}{C} B_1 = -\frac{4\pi ik_p}{c} C DI. \quad (60)$$

From eq. (35) we see that the radial electric field for  $r < a$  is largely due to the term in  $E_1$  since  $k_f \ll k_1$  (at low frequencies). That is,

$$E_r(r < a) \approx -\frac{i}{k_p} E_1 \frac{I'_1(k_p r)}{I_1(k_p a)} e^{-i\phi} e^{i(k_1 z - \omega t)} = -\frac{4\pi C DI}{c} \frac{I'_1(k_p r)}{I_1(k_p a)} e^{-i\phi} e^{i(k_1 z - \omega t)}. \quad (61)$$

The peak voltage difference between the opposing currents is therefore

$$V = 2 \int_0^a |E_r| dr \approx \frac{4\pi}{c} C DI = Z_0 I, \quad (62)$$

where

$$Z_0 \approx 377 CD \Omega. \quad (63)$$

When  $\theta = 45^\circ$ ,

$$D \approx \frac{0.55 \cdot 0.6}{(1.2 - 0.44) \cdot 0.6 + 0.55 \cdot (0.4 + 0.6)} = 0.35, \quad (64)$$

so that

$$Z_0(\theta = 45^\circ) \approx 377 \cdot 1.4 \cdot 0.35 = 185 \Omega. \quad (65)$$

In practice, the wires of the twisted pair are insulated, which reduces the characteristic impedance to  $\approx 100 \Omega$ .

For gentle twists ( $k_p a \ll 1$ ) eq. (59) simplifies to

$$D \approx \frac{I_1(k_p a)}{k_p a I_0(k_p a)} \approx \frac{1}{2}, \quad (66)$$

so that, recalling eq. (50),

$$Z_0(\theta \approx 0) \approx \frac{189}{\sqrt{\cos \theta}} \Omega, \quad (67)$$

little different from the value at  $\theta = 45^\circ$ .

## 2.4 Energy Flux, Momentum and Angular Momentum Density

At low frequencies where  $k'_1 \approx k_1 \approx k_p \gg k_f$  the electromagnetic fields for  $r < a$  follow from eq. (35)-(40) using eqs. (58) and (60) for the constants  $E_1$  and  $B_1$  in terms of the peak current  $I$ ,

$$E_r \approx -\frac{4\pi}{c} \frac{CDI}{2} \frac{I'_1(k_p r)}{I_1(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \quad (68)$$

$$E_\phi \approx \frac{4\pi}{c} \frac{iCDI}{2r} \frac{I_1(k_p r)}{I_1(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \quad (69)$$

$$E_z \approx -\frac{4\pi}{c} \frac{ik_p CDI}{2} \frac{I_1(k_p r)}{I_1(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \quad (70)$$

$$B_r \approx -\frac{4\pi}{c} \frac{iC^2 DI}{2a} \frac{I'_1(k_p r)}{I'_1(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \quad (71)$$

$$B_\phi \approx -\frac{4\pi}{c} \frac{C^2 DI}{2ar} \frac{I_1(k_p r)}{I'_1(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \quad (72)$$

$$B_z \approx \frac{4\pi}{c} \frac{k_p C^2 DI}{2a} \frac{I_1(k_p r)}{I'_1(k_p a)} e^{-i\phi} e^{i(k_p z - \omega t)}, \quad (73)$$

and for  $r > a$  we have the forms (68)-(73) with the substitution  $I_1 \rightarrow K_1$ .

The electric field components (68)-(70) have similar strength (in Gaussian units) to the magnetic field components (71)-(73). The latter correspond to the  $m = 1$  term in the series expansions for the quasistatic magnetic fields given in [5, 6, 7, 8].

The time-average Poynting vector  $\langle \mathbf{S} \rangle$  for  $r < a$  at low frequencies is

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{c}{8\pi} \text{Re}(\mathbf{E} \times \mathbf{B}^*) = \frac{c}{8\pi} \text{Re}[(E_\phi B_z^* - E_z B_\phi^*) \hat{\mathbf{r}} + (E_z B_r^* - E_r B_z^*) \hat{\boldsymbol{\phi}} + (E_r B_\phi^* - E_\phi B_r^*) \hat{\mathbf{z}}] \\ &\approx \frac{4\pi}{c} \frac{C^3 D^2 I^2}{4a} \frac{I_1(k_p r) I'_1(k_p r)}{I_1(k_p a) I'_1(k_p a)} \left[ k_p \hat{\boldsymbol{\phi}} + \frac{\hat{\mathbf{z}}}{r} \right], \end{aligned} \quad (74)$$

and that for  $r > a$  is obtained from eq. (74) with the substitution  $I_1 \rightarrow K_1$ .

At low frequencies there is no time-average flow of energy in the radial direction, and hence no radiation is emitted by the transmission line.<sup>7</sup>

The energy-flow/Poynting vector (74) is in the same direction as the group velocity (53), as generally expected.<sup>8</sup> Lines of the Poynting flux  $\langle \mathbf{S} \rangle$  on the cylinder of radius  $r$  follow helices that make angle

$$\theta_g \approx \tan^{-1} k_p r \quad (54)$$

to the  $z$ -axis, such that only at  $r = a$  does the energy flow in a helix whose angle matches that of the windings,  $\theta$ . At small  $r$  the (small) energy flows largely parallel to the axis. At large  $r$  the angle  $\theta_S$  approaches  $90^\circ$  and the Poynting vector is almost entirely transverse; however because  $K_1(k_p r) \rightarrow 0$  at large  $r$  there is very little energy associated with these very tight spirals.

<sup>7</sup>Even if we keep the smaller terms in  $E_\phi$  and  $B_\phi$  of eqs. (36) and (39) there is still no radiation emitted by the transmission line at low frequencies.

<sup>8</sup>See, for example, sec. 2.1 of [12] and references therein.

The Poynting vector is at right angles to the wave vector (9), whose angle  $\theta_k$  to the  $z$ -axis is given by eq. (12).

The Poynting vector plays the dual role of describing energy flux and momentum density, where the latter is given by

$$\langle \mathbf{p} \rangle = \frac{\langle \mathbf{S} \rangle}{c^2} \quad (75)$$

in vacuum. The density  $\mathbf{l}$  of angular momentum in the electromagnetic field is therefore

$$\langle \mathbf{l} \rangle = \mathbf{r} \times \langle \mathbf{p} \rangle = \mathbf{r} \times \frac{\langle \mathbf{S} \rangle}{c^2}. \quad (76)$$

On averaging over azimuth  $\phi$  only the  $z$ -component of the angular momentum is nonzero,

$$\langle \mathbf{l} \rangle = \frac{4\pi}{c} \frac{C^3 D^2 I^2}{4a} \frac{I_1(k_p r) I_1'(k_p r)}{I_1(k_p a) I_1'(k_p a)} \frac{k_p r}{c^2} \hat{\mathbf{z}}. \quad (77)$$

Thus, the electromagnetic waves on a right-handed twisted-pair transmission line carry positive angular momentum. *In a quantum view, the photons of the wave have angular momentum  $\hbar$  and energy  $\hbar\omega$ . Hence, we expect that  $\langle \mathbf{l} \rangle = (\langle u \rangle / \omega) \hat{\mathbf{z}}$  where  $\langle u \rangle = (|E|^2 + |B|^2) / 8\pi$  is the time-average electromagnetic energy density. However, this relation is not self evident given the description of the waves in terms of Bessel functions.*

## A Appendix: A Single Wire Helix

We can compare the twisted-pair transmission line to the case of a single helical wire [1, 2] in the “sheath” approximation that the helical current flows at angle  $\psi$  uniformly over the entire cylinder  $r = a$ , such that the current and fields have no azimuthal dependence. Then, instead of eqs. (35)-(40)  $r < a$ , we now have

$$E_r = -\frac{ik_1}{k_0'^2} E_0 \frac{I_0'(k_0' r)}{I_0(k_0' a)} e^{i(k_0 z - \omega t)}, \quad (78)$$

$$E_\phi = \frac{ik_f}{k_0'^2} B_0 \frac{I_0'(k_0' r)}{I_0'(k_0' a)} e^{i(k_0 z - \omega t)}, \quad (79)$$

$$E_z = E_0 \frac{I_0(k_0' r)}{I_0(k_0' a)} e^{i(k_0 z - \omega t)}, \quad (80)$$

$$B_r = -\frac{ik_1}{k_1'^2} B_0 \frac{I_0'(k_0' r)}{I_0'(k_0' a)} e^{i(k_0 z - \omega t)}, \quad (81)$$

$$B_\phi = -\frac{ik_f}{k_1'^2} E_0 \frac{I_0'(k_0' r)}{I_0(k_0' a)} e^{i(k_0 z - \omega t)}, \quad (82)$$

$$B_z = B_0 \frac{I_0(k_0' r)}{I_0'(k_0' a)} e^{i(k_0 z - \omega t)}, \quad (83)$$

and for  $r > a$  we have the forms (78)-(83) with the substitution  $I_0 \rightarrow K_0$ .

The condition (41) now implies that

$$k_0'^2 E_0 + ik_f \cot \psi B_0 = 0. \quad (84)$$

Similarly, the condition (43) implies that

$$ik_f \cot \psi I'_0(k'_0 a) K'_0(k'_0 a) E_0 + k_0'^2 I_0(k'_0 a) K_0(k'_0 a) B_0 = 0. \quad (85)$$

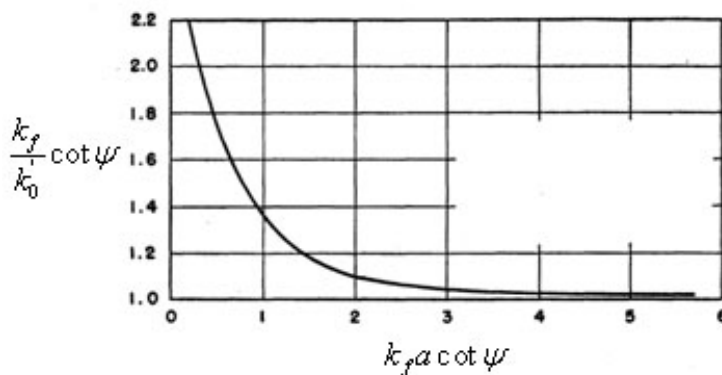
The vanishing of the determinant of the coefficient matrix tells us that

$$k_0'^4 = -k_f^2 \cot^2 \psi \frac{I'_0(k'_0 a) K'_0(k'_0 a)}{I_0(k'_0 a) K_0(k'_0 a)} = k_0'^2 k_f^2 \cot^2 \psi \frac{I_1(k'_0 a) K_1(k'_0 a)}{I_0(k'_0 a) K_0(k'_0 a)}, \quad (86)$$

recalling eqs. (33)-(34). That is,

$$k_0' \sqrt{\frac{I_0(k'_0 a) K_0(k'_0 a)}{I_1(k'_0 a) K_1(k'_0 a)}} = k_f \cot \psi. \quad (87)$$

At low frequencies such that  $k_f a \ll 1$  the factor involving Bessel functions in eq. (87) becomes large, and  $k_0' \ll k_f$ , as illustrated in the figure below, from [1].



Then,  $k_0 = \sqrt{k_f^2 + k_0'^2} \approx k_f$  so that the phase velocity and group velocity are both very close to  $c$ .

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