Two Conducting Spheres at the Same Potential

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1 Problem

The centers of two conducting spheres of radii a and a' are separated by distance $b \gg a + a'$, and the two spheres are at the same electric potential. The latter condition could be enforced by connecting the two spheres with a fine wire. Deduce a relation between charges Q and Q' that reside on spheres, accurate to terms of order a^2/b^2, aa'/b^2 and a'^2/b^2.

Comment also on the case that $b \gtrsim a \gg a'$, which could correspond to "grounding" a conducting sphere of radius a' to the Earth of radius a.

2 Solution

This problem is taken from the classic essay by G. Green (1828) [1], who gave an answer only to first order of accuracy.

To zeroth order, the charges Q and Q' are uniformly distributed on the conducting spheres bring each to the same potential, independent of the presence of the other sphere. That is,

$$\frac{Q}{a} = \frac{Q'}{a'} \qquad \text{(0th order).} \tag{1}$$

To first order, the potential at the center of each sphere can be taken as that due to sum of the potentials due to uniform charge distributions on the spheres,

$$\frac{Q}{a} + \frac{Q'}{b} = \frac{Q'}{a'} + \frac{Q}{b}\,, \tag{2}$$

and hence,

$$\frac{Q}{a}\left(1 - \frac{a}{b}\right) = \frac{Q'}{a'}\left(1 - \frac{a'}{b}\right) \qquad \text{(1st order).} \tag{3}$$

In general, the charge distributions on the spheres are not uniform, due to their mutual electrical influence. A solution accurate to any desired order can be obtain by use of the image method, in which the charge distribution of the spheres is represented by a sequence of point charges at appropriate locations along their line of centers. We write

$$Q = Q_0 + Q_1 + Q_2 + ..., \qquad Q' = Q'_0 + Q'_1 + Q'_2 + ..., \tag{4}$$

where charges Q_0 and Q'_0 are located at the centers of the two spheres. To keep the second sphere at an equipotential under the influence of charge Q_0 on the first, we follow the usual prescription in placing charge

$$Q'_1 = -Q_0 \frac{a'}{b} \qquad \text{at} \qquad r'_1 = \frac{a'^2}{b} \tag{5}$$

from the center of the second sphere. Likewise, to keep the first sphere at an equipotential, we place charge

$$Q_1 = -Q'_0 \frac{a}{b} \qquad \text{at} \qquad r_1 = \frac{a^2}{b} \tag{6}$$

from the center of that sphere. Then, to keep the spheres at equipotentials under the influence of charges Q_1 and Q'_1, we add charges

$$Q'_2 = -Q_1 \frac{a'}{b - r_1} = Q'_0 \frac{aa'}{b^2 - a^2} \qquad \text{at} \qquad r'_2 = \frac{{a'}^2}{b - r_1}\,, \tag{7}$$

and

$$Q_2 = -Q'_1 \frac{a}{b - r'_1} = Q_0 \frac{aa'}{b^2 - a'^2} \qquad \text{at} \qquad r_2 = \frac{a^2}{b - r'_1}\,, \tag{8}$$

etc.

The additional charges Q_1, Q_2, ..., Q'_1, Q'_2, ..., have been positioned so that they do not change the potentials of the two spheres. The condition that the two spheres be at the same potential is therefore

$$\frac{Q_0}{a} = \frac{Q'_0}{a'}\,. \tag{9}$$

To 2nd order, the total charge on the first sphere can now be written as

$$Q = Q_0 + Q_1 + Q_2 = Q_0 - \frac{Q'_0 a}{b} + Q_0 \frac{aa'}{b^2 - a'^2} = Q_0 \left(1 - \frac{a'}{b} + \frac{aa'}{b^2 - a'^2}\right), \tag{10}$$

while

$$Q' = Q'_0 + Q'_1 + Q'_2 = Q'_0 - \frac{Q_0 a'}{b} + Q'_0 \frac{aa'}{b^2 - a^2} = Q'_0 \left(1 - \frac{a}{b} + \frac{aa'}{b^2 - a^2}\right). \tag{11}$$

We eliminate Q_0 and Q'_0 by using eq. (9) again to find (for any $b > a + a'$)

$$\frac{Q}{a}\left(1 - \frac{a}{b} + \frac{aa'}{b^2 - a^2}\right) = \frac{Q'}{a'}\left(1 - \frac{a'}{b} + \frac{aa'}{b^2 - a'^2}\right). \tag{12}$$

The result (12) was first obtained by Thomson [2]. See also sec. 174 of [3], from which the expansion to fifth order can be inferred. At third order, eq. (12) becomes

$$\frac{Q}{a}\left(1 - \frac{a}{b} + \frac{aa'}{b^2 - a^2} - \frac{a^2 a'}{b(b^2 - a^2 - a'^2)}\right) = \frac{Q'}{a'}\left(1 - \frac{a'}{b} + \frac{aa'}{b^2 - a'^2} - \frac{aa'^2}{b(b^2 - a^2 - a'^2)}\right). \tag{13}$$

To 2nd order of accuracy when $b \gg a + a'$ this can also be written as

$$\frac{Q}{a}\left(1 - \frac{a}{b} + \frac{aa'}{b^2}\right) = \frac{Q'}{a'}\left(1 - \frac{a'}{b} + \frac{aa'}{b^2}\right). \tag{14}$$

The 1st-order result (3) is contained within the 2nd-order result (14), as expected.

2.1 "Grounding:" $b = a + d$ with $a \gg d \gg a'$

This section added Sept. 5, 2011.

When $a \gg d \gg a'$, the electric field due to the large sphere in the vicinity of the small sphere (and in its absence) is essentially uniform with value $E_a = Q/a^2$. The potential due to charge Q at distance d above the surface of the large sphere is lower that the surface potential Q/a by $\Delta V \approx E_a d = Qd/a$. When the small sphere is at distance d, placing charge Q' on it raises its potential by Q'/a', so with charge

$$Q' \approx \frac{a'd}{a^2} Q, \tag{15}$$

the small sphere has the same potential as the large one.

Surprisingly, this result does not follow from the second-order image result eq. (12), in that the terms in parenthesis on the left can be approximated as $d/a + a'/2d$ and the terms in parenthesis on the right by 1, to find

$$Q' \approx \frac{a'}{a} \left(\frac{d}{a} + \frac{a'}{2d} \right) Q \qquad \text{(second order)}. \tag{16}$$

However, when we use the third-order result (13) the (relatively large) term $a'/2d$ in eq. (16) is cancelled and we obtain eq. (15).

For example, if the first sphere is the Earth with raidus $a \approx 6.4 \times 10^8$ cm and the small sphere has radius $a' = 1$ cm at distance $d = 10$ m above the Earth, then "grounding" the small sphere to Earth via a fine wire would leave it with charge

$$Q' \approx \frac{a'd}{a^2} Q \approx -\frac{1 \cdot 10^3 \cdot 5 \times 10^5}{4 \times 10^{17}} \approx -10^{-9} \text{ C}, \tag{17}$$

noting that the electric charge Q of the Earth is about $-500,000$ C [4].

"Grounding" a conductor does not reduce its charge to zero, but only to a practically negligible amount.

2.2 Two Conducting Spheres in Contact

A solution by the method of inversion is given in sec. 175 of [3] and by the method of images in [5]. For $a' \ll a$ the charges on the two conducting spheres, in contact, are related by

$$Q' \approx \frac{\pi^2 a'^2}{6a^2} Q = 1.65 \frac{a'^2}{a^2} Q, \tag{18}$$

which is slightly larger than the prediction of eq. (15) when $d = a'$.

References

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